Monograph

Simplified Reproducing Kernel Space Theory and Its Applications

LIANGCAI MEI

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Introduction

In the field of science and engineering research, it is difficult to obtain the analytical solutions of many differential equations, so it is particularly necessary for researchers to design effective numerical solutions to differential equations. In order to facilitate the study of these different forms of differential equations, scholars express these differential equations as operators in a specific space. Different problems correspond to different operator structures, and designing effective numerical algorithms for operator equations has become an important means to solve problems.

Reproducing kernel space is a field in functional analysis, which is a special Hilbert space since its proposal in the 1980s, scholars have become active in studying the application of the theory of reproducing kernel and related methods. People have found that it can be applied in many aspects, such as signal processing, numerical solutions of differential equations, image processing, etc. In recent years, there have been many international literature discussing the application prospects of regenerative kernels.

The simplified reproducing kernel method avoids the process of Schmidt orthogonalization to obtain spatial bases, and has advantages such as spatial reproducibility and controllable regularity. It has become one of the main directions for scholars to study the theory of reproducing kernels in recent years. As an important mathematical tool in scientific and engineering calculations, the Regenerative Kernel Method is mainly studied in this book for several special numerical algorithms of differential equations. It provides a detailed analysis and introduction to the algorithm principle, stability, convergence, error analysis, applicability, and advantages and disadvantages of the simplified Regenerative Kernel Method. The publication of this academic monograph can provide readers with a deeper and more accurate understanding of the theory of regenerative nuclei.

I was able to complete the writing of this book without the help of my teacher and workplace. Firstly, I would like to express my special gratitude to my research supervisor Professor Lin Yingzhen. Professor Lin has led me into the academic arena, allowing me to appreciate the charm of mathematics. His rigorous thinking and teaching spirit have deeply influenced me. Secondly, I would like to express my gratitude to my employer-Zhuhai College, Beijing Institute of Technology. The publication of this book is inseparable from the support and funding of the institution. The book was jointly funded by the Guangdong Provincial Department of Education's General University Characteristic

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Due to the limited research level of the author, there must be many shortcomings and errors in this book. We sincerely request readers to correct them for further revisions in the future.

Liangcai Mei
Autumn 2023

Chapter 1 Introduction to reproducing kernel method

1.1 The background and significance of writing

Differential equations are one of the most important mathematical methods and tools for describing the basic laws of physics, and have important significance in scientific and engineering research. They are commonly used to model complex systems under non ideal conditions, such as fluid motion, wave propagation, and price evolution of stocks and options. However, analytical solutions to most differential equations are difficult to obtain, so researchers generally consider obtaining approximate solutions to the equations.

Although the research field of numerical solutions for differential equations is very active and full of lasting vitality, there are still many differential equations in the field of technology that cannot obtain analytical solutions and good approximate solutions. On the other hand, the accuracy, convergence order, stability, and computational efficiency of numerical solutions for differential equations still require continuous exploration by scholars. How to design numerical algorithms for differential equations with high accuracy and strong stability is still one of the hot topics studied by scholars.

Differential equations include various forms such as ordinary differential equations, partial differential equations, calculus equations, fractional differential equations, etc. The definite solution conditions also include various mixed forms such as multi-point boundary values, integral boundary values, and differential boundary values. In order to facilitate the study of these different forms of differential equations, scholars have summarized these equations into a unified operator equation form in different spaces. In the field of engineering applications, different natural phenomena can be characterized by different manifestations of linear and nonlinear operators in operator equations. Therefore, the theoretical analysis and numerical solution of operator equations have a wide range of practical application backgrounds, whether in mathematical research fields such as functional analysis and numerical analysis, or in technological engineering applications such as geophysical inversion, non fluid mechanics, and ultrasonic testing, the study of numerical solutions of operator equations has a very important position and a wide research space.

From the perspective of practical applications, the study of numerical solutions for operator equations is necessary in the field of natural science applications. For example, in the field of biology, in a single population differential equation model, if an approximate solution can be obtained using numerical methods, it can more accurately reflect the changes in population size over time. The numerical solution of the predator-prey model can approximate the changes between prey and predator, and these changes are beneficial for humans to explore the mysteries of nature more efficiently.

From the perspective of numerical solution methods for differential equations, different numerical solutions for operator equations are beneficial for improving the approximation and stability of approximate solutions. The commonly used research methods for numerical solutions of operator equations include variational methods, finite difference methods, etc. Based on the specific forms of operator equations, scholars are gradually proposing various numerical solutions. Especially since the 1980s, applied reproducing kernel spaces have been proposed, and various reproducing kernel solving algorithms have been developed both domestically and internationally. They have also solved some linear and nonlinear operator equation numerical algorithm problems.

From the perspective of the development of mathematics itself, the proposal of new solutions to operator equations is conducive to promoting the further development of mathematical theory in functional analysis, differential equations, scientific calculations, and other fields. The concept of reproducing kernel was proposed by scholar Zaremba in 1908. After more than a hundred years of development, the theory of reproducing kernel has continuously developed as a new mathematical theory for solving differential and integral operator equations. In the field of scientific computing, the theory of reproducing kernel has attracted the attention of many scholars due to its computational advantages of reproducing.

The reproducing kernel method is named after its reproducing property in spatial inner product operations. Compared with other numerical methods, the reproducing kernel method has the characteristic of regularity and controllability, which is very advantageous for solving special physical models with limited smoothness of solutions. In addition, the reproducing kernel method can flexibly construct a reproducing kernel space that satisfies the definite solution conditions when solving differential equations, so that the constructed space can describe some properties of the definite solution, which is very advantageous for characterizing the characteristics of the model.

In summary, considering the wide application background of numerical solutions for operator equations and the theory of reproducing kernels, this book proposes some new numerical algorithms based on the simplified theory of reproducing kernel space, and conducts numerical solution research on some operator equations. Therefore, this book has certain academic value both in terms of fundamental theory and practical application.

1.2 The generation and development of reproducing kernel theory

reproducing kernel space is a unique type of Hilbert function space, named by scholars due to the existence of a reproducible kernel function $R(x, y)$ in the space. Any function $f(x)$ has a reproducible property in the form of Eq. (1-1) in the spatial product operation, which brings convenience to calculations. In the early 19th century, Bergmann[1] first introduced the concept and expression of the reproducing kernel, forming the definition from space to inner product and then to reproducing kernel. The reproducing kernel was also developed as a computational method for solving operator equations.

$$f(y) = \langle f(x), R(x, y)\rangle \tag{1-1}$$

(1) Reproducing kernel in segmented exponential stage

In 1986, Cui[2] proposed the reproducing kernel space $W_2^1[a, b]$ and its piecewise exponential expression of the kernel function, which is a landmark starting point in the study of reproducing kernel theory, opening up a new direction for numerical analysis and laying a theoretical framework for the subsequent use of reproducing kernels in numerical analysis.

The regenerated kernel space $W_2^1[a, b]$ is as follows

$$W_2^1[a, b] = \{u(x)\,|u(x)\,\text{is is absolutely continuous on}[a, b], u'(x) \in L^2[a, b]\}$$

and its inner product is

$$\langle u(x), v(x)\rangle_{W_2^1} = \int_a^b u(x)v(x)dx + \int_a^b u'(x)v'(x)dx, \quad u, v \in W_2^1[a, b]$$

In addition, to meet the needs of numerical calculations, Cui[3] also provided an expression for the reproducing kernel space $W_2^m[a, b]$ with a piecewise exponential form

$$R(x, y) = \begin{cases} \sum\limits_{i=0}^{m} c_i(y)e^{ix}, & x \le y \\ \sum\limits_{i=0}^{m} d_i(y)e^{-ix}, & x > y \end{cases} \tag{1-2}$$

Based on the piecewise exponential form (Eq. (1-2)) of reproducing kernel, Wu[4], Yan[5], Li[6], and others have provided various types of reproducing kernel spaces and their kernel functions, which are used to numerically solve some operator equations. Although the reproducing kernel in the segmented exponential stage can be represented by an analytical function, the e in Eq. (1-2) is an irrational number, which leads to shortcomings such as high computational complexity, long processing time, and high computational speed. Scholars have to seek a more concise way to construct the reproducing kernel.

(2) Reproducing kernel in segmented polynomial stage

In 2008, Wu, Lin, and others provided a unified inner product form in the reproducing kernel space $H_2^m[a,b]$ and obtained the piecewise polynomial form of the reproducing kernel, marking the starting point of research on piecewise polynomial reproducing kernels.

$$H_2^m[a,b] = \{u(x)|u^{(m-1)}(x)\text{is is absolutely continuous on}[a,b], u_m(x) \in L_2[a,b]\}$$

and its inner product is

$$\langle u(x), v(x)\rangle_{H_2^m} = \sum_{i=0}^{m-1} u_i(a)v_i(a) + \int_a^b u^{(m)}(x)v^{(m)}(x)dx$$

Wu and Lin[7] authored "Applied Reproducing Kernel Space", in which they fully proved that reproducing kernel space $H_2^m[a,b]$ is equivalent to reproducing kernel space $W_2^m[a,b]$ given by Cui. From a numerical perspective, using polynomial producing kernel in space $H_2^m[a,b]$ can compensate for the shortcomings of exponential reproducing kernel. In addition, Wu and Lin[7] systematically introduced the construction process of polynomial reproducing kernels in their compilation, and applied the construction idea to many reproducing kernel spaces, such as periodic boundary value reproducing kernel spaces, integral boundary value reproducing kernel spaces, weighted reproducing kernel spaces, etc. They solved various types

$$R(x,y) = \begin{cases} \sum_{i=0}^{m} c_i(y)x^i, & x \le y \\ \sum_{i=0}^{m} d_i(y)x^i, & x > y \end{cases} \tag{1-3}$$

Since $R(x,y)$ is in the form of polynomials in different situations, theoretically Eq. (1-3) is the simplest type of reproducing kernel, and the computational complexity is greatly reduced compared to exponential reproducing kernels. Scholars have begun to use polynomial kernels to solve more complex initial boundary value problems[8,9].

Jia[10] et al. obtained a unified expression for the regeneration kernel space $H_2^m[a,b]$

and obtained several properties related to the regeneration kernel. Geng[11], Niu[12], and Li [13] used a combination of polynomial regeneration kernel method and other methods to solve various operator equations.

The exact solution of the equation obtained by the piecewise polynomial reproducing kernel method generally has the expression form Eq. (1-4) .

$$u(x) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} f(x_k) \widetilde{\psi}_i(x) \tag{1-4}$$

In the exact solution $u(x)$, β_{ik} is the orthogonalization coefficient, and $\{\widetilde{\psi}_i\}_{i=1}^{\infty}$ is the standard orthogonal system after Schmidt orthogonalization. Wu[7] et al. provided a detailed description of the theoretical properties and algorithm steps related to Schmidt orthogonalization in the literature. The calculation of $\widetilde{\psi}_i$ mainly relies on the following formula

$$\widetilde{\psi}_i = \frac{\psi_i - \sum\limits_{k=1}^{i-1} \langle \psi_i, \widetilde{\psi}_i \rangle \widetilde{\psi}_i}{\|\psi_i - \sum\limits_{k=1}^{i-1} \langle \psi_i, \widetilde{\psi}_i \rangle \widetilde{\psi}_i\|}$$

From the above equation, it can be seen that in the Schmidt orthogonalization process, calculating each function $\widetilde{\psi}_i(x)$ requires multiple operations such as inner product, product, and sum, and involves quotient operations with norm. It can be seen that Schmidt orthogonalization is a very tedious process. This method of constructing standard orthogonal bases through Schmidt orthogonalization is slightly more complex, resulting in long computational time for solving high-order problems, and even inability to calculate ideal results.

(3) Research progress on simplified regeneration accounting method

Considering the complexity of Schmidt orthogonalization calculation process, in recent years, Lin[14] has constructed a simplified reproducing kernel method, which avoids the calculation process of Schmidt orthogonalization and cleverly utilizes the reproducing property of the reproducing kernel to obtain numerical solutions of differential equations by solving linear equations. Compared to the traditional reproducing kernel method, this algorithm has advantages such as bounded reproducing kernels, intuitive algorithms, and easy programming experiments. Based on this simplified reproducing kernel method, many scholars have studied the numerical solutions of various operator equations through different reproducing kernel spaces. The exact solutions obtained from the simplified reproducing kernel space generally take the form of Eq. (1-5) .

$$u(x) = \sum_{k=1}^{\infty} c_i \psi_i(x) \tag{1-5}$$

Zhao[15] solved a numerical solution problem for a class of high-order linear differential Eq. (1-6) using the simplified reproducing kernel method in space $W_2^m[a,b]$, and proposed several property theories for the convergence order of the algorithm.

$$\begin{cases} u^{(m-1)}(x) + a_1(x)u^{(m-2)}(x) + \cdots + a_{m-1}(x)u(x) = f(x) \\ T_1 u = 0, T_2 u = 0, \ \cdots, T_{m-1}u = 0 \end{cases} \tag{1-6}$$

Xu[16] et al. proposed a simplified reproducing kernel method to numerically solve the one-dimensional elliptical interface Eq. (1-7) , which is actually a second-order linear differential equation with discontinuous coefficients. The author also demonstrated the stability of the algorithm.

$$\begin{cases} (\beta_1 u')' - \gamma_1 u = f_1(x), 0 < x < \alpha \\ (\beta_2 u')' - \gamma_2 u = f_2(x), \alpha < x < 1 \\ u(0) = a \ \ u(1) = b \end{cases} \tag{1-7}$$

Eq. (1-7) satisfies the following interface conditions

$$\begin{cases} [u]|_\alpha = u(\alpha^+) - u(\alpha^-) = 0 \\ [\beta u']|_\alpha = \beta_2(\alpha^+)u(\alpha^+) - \beta_1(\alpha^-)u'^{(\alpha^-)} = 0 \end{cases}$$

Xu[17, 18] used the simplified reproducing kernel method to numerically solve the nonlinear fractional order model and the nonlinear fractional order delay model, respectively. In addition, Niu[19] used the reproducing kernel method to solve several types of nonlinear singular boundary value problems.

In addition, with the proposal of various research methods, scholars are increasingly focusing on the idea of integrating various algorithm theories to solve problems, and leveraging the advantages of various algorithms to construct new algorithms has become a new research trend. Zhang[20] and Zheng[21] have proposed various numerical solutions for solving linear operator equations by combining multi-scale orthogonality and compactness with simplified reproducing kernel methods.

In recent years, the theory of reproducing kernel space has not only received attention from scholars in the field of numerical analysis, but has also made significant progress in application fields. The theory of reproducing kernel has been successfully applied in image processing, signal analysis, machine learning, pattern recognition, and other fields. The theory of reproducing kernels is of great significance for exploring natural laws, especially in today's rapidly developing artificial intelligence and big data technologies. The research of reproducing kernels in fields such as machine learning and deep data mining has also attracted the attention of many scholars. Hao[22] et al. combined the multi-scale reproducing kernel method with convolutional neural networks and applied it to single-cell image classification problems. This method can be widely applied in artificial intelligence fields such as facial recognition.

1.3 The main work of this book

The main content and structure of this book are as follows:

In chapter 1, firstly, outlines the background and research significance of the writing of this book. Secondly, the writing foundation of this book is introduced from the introduction of operator equations and the current research status of reproducing kernel theory. Finally, the main work of writing this book is summarized.

In chapter 2, the definition and properties of the reproducing kernel space are first introduced, and the general form of the new reproducing kernel space $W^m[a,b]$ and its kernel function $R(x,y)$ is given through lemma. Secondly, the construction principle of discontinuous regenerative kernel space is presented.

In chapter 3, we have considered the reproducing kernel method for second-order impulsive differential equations as follows

$$\begin{cases} u''(x) + a_1(x)u'(x) + a_0(x)u(x) = f(x), \quad x \in (a,b) \setminus \{c\} \\ u(a) = \alpha_1, \ u(b) = \alpha_2 \\ \Delta u'(c) = \alpha_3, \ \Delta u(c) = \alpha_4 \end{cases}$$

where, $\Delta u'(c) = u'(c^+) - u'(c^-)$, α_3 and α_4 are not at the same time as 0. $a_i(x)$ and $f(x)$ are known function, $\alpha_j \in \mathbb{R}$, $j = 1,2,3,4$. The main idea of this method is to establish a non-smooth reproducing kernel space that can be used in pulse models. And the uniform convergence of the numerical solution is proved, the time consuming Schmidt orthogonalization process is avoided. The algorithm is proved to be feasible and effective through some numerical examples.

In chapter 4, we consider the reproducing kernel method for the following second-order nonlinear impulsive differential equations

$$\begin{cases} u''(x) + a_1(x)u'(x) + a_0(x)u(x) + \mathcal{N}(u) = f(x), & x \in [a,b] \setminus \{c\} \\ u(a) = \alpha_1, u(b) = \alpha_2, \Delta u'(c) = \alpha_3, \Delta u(c) = \alpha_4 \end{cases}$$

where, $\Delta u'(c) = u'(c^+) - u'(c^-)$, α_3 and α_4 are not at the same time as 0. $a_i(x)$ and $f(x)$ are known function, $\mathcal{N} : \mathbb{R} \to \mathbb{R}$ is a continuous function, $\alpha_j \in \mathbb{R}, j = 1, 2, 3, 4$. This method combines the reproducing kernel method with the least squares method to solve the second-order nonlinear impulsive differential equations. Then the uniform convergence of the numerical solution is proved, and the time consuming Schmidt orthogonalization process is avoided. The algorithm is employed successfully on some numerical examples.

In chapter 5, the simplified reproducing kernel method is developed to obtain stable numerical solutions of second-order boundary value problems of the following Fredholm integro-differential equation.

$$\begin{cases} u''(s) + p(s)u'(s) + q(s)u(s) + \lambda \int_0^1 k(s,t)u(t)dt = f(s), & s \in [0,1], \\ u(0) = \alpha, \quad u(1) = \beta. \end{cases}$$

where, λ, α, β are real constants, $p(s), q(s)$ are two known functions, $f(s) \in L^2([0,1])$ and $k(s,t) \in C([0,1]\times[0,1])$ are given, and $u(s)$ is an unknown function to be determined. The convergence analysis of the method and the condition number of the matrix are also discussed. The proposed method is proved to be stable and have second order convergence. The algorithm is employed successfully in some numerical examples.

In chapter 6, based on the reproducing kernel direct sum space, the numerical solutions of linear Volterra integral equations with varying coefficients are studied

$$\begin{cases} a_{11}(x)f_1(x) - b_{11}\int_0^x k_{11}(x,t)f_1(t)dt + a_{12}(x)f_2(x) - b_{12}\int_0^x k_{12}(x,t)f_2(t)dt = u_1(x) \\ \\ a_{21}(x)f_1(x) - b_{21}\int_0^x k_{21}(x,t)f_1(t)dt + a_{22}(x)f_2(x) - b_{22}\int_0^x k_{22}(x,t)f_2(t)dt = u_2(x) \end{cases}$$

where, $a_{ij}(x), i, j = 1, 2$ are smooth functions defined on [0,1], and $b_{ij}, i, j = 1, 2$ are given constants. This chapter establishes a reproducing kernel direct sum space that can be used for integral operator equations based on the simplified reproducing kernel method. The convergence order calculation formula is defined based on Euclidean distance, and the convergence order and stability of the numerical algorithm are analyzed. The stability of the method is proved, and it is not less than second-order convergence.

Chapter 2 Basic theory of reproducing kernel space

This chapter mainly introduces the basic knowledge about reproducing kernel spaces, provides the construction process and basic properties of discontinuous reproducing kernel spaces, and provides a knowledge overview for the algorithm construction in subsequent chapters.

2.1 Reproducing kernel space

Lemma 2.1 [7] If W is a Hilbert space and the elements in W are complex valued functions defined on the set X, then the following two propositions are equivalent:

(1) $\forall x \in X$, there is a positive number C_x, so that

$$|f(x)| \leq C_x \|f\|, \quad f \in H$$

(2) $\forall x \in X$, there exists a unique function $K_x(y) \in H$ such that

$$\langle f, K_x \rangle = f(x), \quad f \in H$$

Definition 2.1 [7] If Hilbert space W satisfies the lemma 2.1 , then W is called the reproducing kernel space, and the binary function

$$K(x, y) \triangleq K_y(x)$$

is called the reproducing kernel of W.

Some basic properties of reproducing kernel functions and reproducing kernel spaces, such as uniqueness and conjugate symmetry, can be found in ref.[7] or related literature on reproducing kernel. Below are two very classic lemmas about simplified reproducing kernel methods.

Lemma 2.2 [7]

$$W_2^m[a, b] = \{u(x)|u^{(m-1)}(x) \text{ is absolutely continuous on}[a, b], u^{(m)}(x) \in L^2(a, b)\}$$

the space $W_2^m[a, b]$ has the following inner product form

$$\langle u, v \rangle_{W_2^m} = \sum_{k=0}^{m-1} u^{(k)}(a)v^{(k)}(a) + \int_a^b u^{(m)}(x)v^{(m)}(x)dx, \qquad u, v \in W_2^m[a, b]$$

and the norm is defined as

$$\|u\|_{W_2^m} = \sqrt{\langle u, u \rangle_{W_2^m}}, \qquad u \in W_2^m[a, b]$$

then

(1) $W_2^m[a,b]$is a Hilbert space;

(2) $W_2^m[a,b]$is a reproducing kernel space.

Lemma 2.3 [10] The reproducing kernel funciton of $W_2^m[a,b]$ is

$$R_y(x) \triangleq R(x,y) = \begin{cases} \sum_{i=0}^{m-1} (\frac{(x-a)^i}{i!} + (-1)^{m-1-i}\frac{(x-a)^{2m-1-i}}{(2m-1-i)!})\frac{(y-a)^i}{i!}, & x \le y \\ \sum_{i=0}^{m-1} (\frac{(y-a)^i}{i!} + (-1)^{m-1-i}\frac{(y-a)^{2m-1-i}}{(2m-1-i)!})\frac{(x-a)^i}{i!}, & x > y \end{cases} \tag{2-1}$$

Lemma 2.2 and lemma 2.3 provide a new definition of reproducing kernel space and a general expression (Eq. (2-1)) of the reproducing kernel function $R(x,y)$. The relevant conclusions will play an important role in the subsequent chapters of this book.

2.2 Discontinuous reproducing kernel space

From Eq. (2-1) , it can be seen that when using traditional reproducing kernel s-pace for numerical calculations, it is required that the true or approximate solution of the problem must be smooth. However, in the field of engineering research, there are many instances of unsmooth model true solutions and their derivatives, and even intermittent phenomena at local points, known as pulse phenomena.

Pulse phenomenon is also a type of intermittent problem. In solving problems containing intermittent phenomena, Reed and Hill proposed the discontinuous finite elemen-t method as early as 1973 to solve neutron transport equation problems[23]. In 1974, Lesaint[24] applied the discontinuous finite element method to solve ordinary differential equations. The proposed method adopted different mesh forms for different elements and had the ability to flexibly handle the discontinuity at the intersection of each elemen-t. Subsequently, the discontinuous finite element method gradually gained widespread attention from scholars in dealing with the problem of piecewise discontinuity[25].

The traditional reproducing kernel method cannot directly solve the pulse problem. In this chapter, using the idea of discontinuous finite element and ingeniously designing, the pulse point is used as a bridge to connect two or more spaces, constructing a space with piecewise smooth characteristics, which is called the discontinuous reproducing kernel space.

Assuming there is a pulse point c on the interval $[a,b]$, and a reproducing kernel space $W_2^3[a,c]$ (W_a^3 for short)、 $W_2^1[a,c]$ (W_a^1 for short) and its inner product is given by the lemma 2.2 . The reproducing kernel functions of W_a^3 and W_a^1 are $R_t^0(x)$ and $r_t^0(x)$, respectively. Similarly, the reproducing kernel space $W_2^3[c,b]$(W_b^3 for short) and $W_2^1[c,b]$(W_b^1 for short) can be obtained its reproducing kernel functions $R_t^1(x)$ and $r_t^1(x)$, respectively.

Definition 2.2 The linear space $W_{2,c}^3$ is defined as

$$W_{2,c}^3[a,b] = \{u(x)|u(x) = u_0(x),\quad x \in [a,c);\quad u(x) = u_1(x),\quad x \in (c,b]\}$$

where, $u_0(x) \in W_a^3, u_1(x) \in W_b^3$. The inner product and norm are defined as

$$\langle u,v\rangle_{W_{2,c}^3} = \langle u_0,v_0\rangle_{W_a^3} + \langle u_1,v_1\rangle_{W_b^3}, \qquad u,v \in W_{2,c}^3[a,b]$$

$$\|u\|_{W_{2,c}^3} = \sqrt{\langle u,u\rangle_{W_{2,c}^3}}, \qquad u \in W_{2,c}^3[a,b]$$

In summary, for each $u(x) \in W_{2,c}^3[a,b]$ has the following form

$$u(x) = \begin{cases} u_0(x), & x < c \\ u_1(x), & x \geq c \end{cases}$$

where, $u_0(x) \in W_a^3, u_1(x) \in W_b^3$.

Theorem 2.1 $W_{2,c}^3[a,b]$ is a inner space.

Proof: Here we only prove that the distributive law operation is satisfied, and other inner product definition conditions can be similarly proven.

$\forall u,v \in W_{2,c}^3[a,b]$

$$\begin{aligned}\langle u+v,w\rangle_{W_{2,c}^3} &= \langle u_0+v_0,w_0\rangle_{W_a^3} + \langle u_1+v_1,w_1\rangle_{W_b^3} \\ &= \langle u_0,w_0\rangle_{W_a^3} + \langle v_0,w_0\rangle_{W_a^3} + \langle u_1,w_1\rangle_{W_b^3} + \langle v_1,w_1\rangle_{W_b^3} \\ &= \langle u,w\rangle_{W_{2,c}^3} + \langle v,w\rangle_{W_{2,c}^3}\end{aligned}$$

□

Theorem 2.2 The space $W_{2,c}^3[a,b]$ is a Hilbert space.

Proof: Suppose that the sequence $\{u_n(x)\}_{n=1}^{\infty}$ is a Cauchy column in $W_{2,c}^3[a,b]$, so,

$$u_n(x) = \begin{cases} u_{0,n}(x), & x < c \\ u_{1,n}(x), & x \geq c \end{cases} \quad n = 1,2,\cdots$$

where $\{u_{0,n}(x)\}_{n=1}^{\infty}$ and $\{u_{1,n}(x)\}_{n=1}^{\infty}$ are Cauchy columns in W_a^3 and W_b^3, respectively.

So, there are two funcitions $g_0(x) \in W_a^3$, $g_1(x) \in W_b^3$, make

$$\|u_{0,n}(x) - g_0(x)\|_{W_a^3}^2 \to 0, \qquad \|u_{1,n}(x) - g_1(x)\|_{W_b^3}^2 \to 0$$

Let

$$g(x) = \begin{cases} g_0(x), & x < c \\ g_1(x), & x \geq c \end{cases}$$

By the definition 2.2 , $g(x) \in W^3_{2,c}[a,b]$, and

$$\|u_n(x) - g(x)\|^2_{W^3_{2,c}} = \|u_{0,n}(x) - g_0(x)\|^2_{W^3_a} + \|u_{1,n}(x) - g_1(x)\|^2_{W^3_b} \to 0$$

So, the space $W^3_{2,c}[a,b]$ is a Hilbert space. □

Theorem 2.3 The space $W^3_{2,c}[a,b]$ is a reproducing kernel space with the reproducing kernel function

$$R_t(x) = \begin{cases} R^0_t(x), & (x,t) \in [a,c) \times [a,c) \\ R^1_t(x), & (x,t) \in [c,b] \times [c,b] \\ 0, & \text{others} \end{cases} \tag{2-2}$$

Proof: Consider arbitrary $u(x) \in W^3_{2,c}[a,b]$.

If $t \in [a,c)$, then

$$\begin{aligned} \langle u(x), R_t(x)\rangle_{W^3_{2,c}} &= \langle u_0(x), R^0_t(x)\rangle_{W^3_a} + \langle u_1(x), 0\rangle_{W^3_b} \\ &= u_0(t) \end{aligned}$$

If $t \in [c,b]$, then

$$\begin{aligned} \langle u(x), R_t(x)\rangle_{W^3_{2,c}} &= \langle u_0(x), 0\rangle_{W^3_a} + \langle u_1(x), R^1_t(x)\rangle_{W^3_b} \\ &= u_1(t) \end{aligned}$$

In conclusions, for every $u(x) \in W^3_{2,c}[a,b]$, it follows that

$$\langle u(x), R_t(x)\rangle_{W^3_{2,c}} = u(t)$$

So, the space $W^3_{2,c}[a,b]$ is a reproducing kernel space with the reproducing kernel function in Eq. (2-2) . □

Similarly, reproducing kernel space $W^1_{2,c}[a,b]$ is defined as

$$W^1_{2,c}[a,b] = \{u(x) | u_0(x) \in W^1_a, u_1(x) \in W^1_b\}$$

And $W^1_{2,c}$ has the following reproducing kernel function

$$r_t(x) = \begin{cases} r^0_t(x), & (x,t) \in [a,c) \times [a,c) \\ r^1_t(x), & (x,t) \in [c,b] \times [c,b] \\ 0, & \text{others} \end{cases}$$

In summary, the constructed space $W^3_{2,c}[a,b]$,$W^1_{2,c}[a,b]$ retains the advantages of inner product calculation satisfying regeneration, while also possessing the property of sharded smoothness. This book refers to this type of reproducing kernel space with sharded smoothness as a discontinuous reproducing kernel space.

2.3 Summary

This chapter presents the definition and related properties of the general reproducing kernel space $W^m_2[a,b]$, and the general definition, inner product, and norm expression of the space are given, and the general expression of the piecewise polynomial reproducing kernel function $R(x,y)$ is also given. In addition, a discontinuous reproducing kernel space has been defined, and the relevant conclusions provide theoretical support for subsequent chapters.

Chapter 3 The reproducing kernel method for impulsive differential equations

Due to the piecewise smooth nature of its true solutions, traditional numerical methods are difficult to directly apply to solving impulsive differential equations. The simplified reproducing kernel method has many advantages, such as the ability to obtain high-precision analytical solutions and easy algorithm implementation. To achieve the reproducing kernel method for pulse problems, it is urgent to solve the key problem: how to construct a Hilbert space with piecewise smooth characteristics. This chapter is based on a simplified reproducing kernel space and solves impulsive differential equations (IDEs for short). By constructing a discontinuous reproducing kernel space with piecewise smoothness and introducing operator equations, a new numerical method is proposed. The reproducing kernel method, which originally could only solve smooth problems, was cleverly applied to solve pulse problems, and an algorithm for solving second-order impulsive differential equations was designed.

3.1 Second-order impulsive differential equation

Pulse boundary value problems occur in many applications: population dynamics[26], physics, chemistry[27], irregular geometries and interface problems[28-30], signal processing[31]. The research on the impulsive differential equations with all kinds of boundary value is much more active in recent years. However, only in the last few decades has the attention been paid to the theory and numerical analysis of IDEs. All kinds of methods have been widely used to study the existence of solutions for impulsive problems[32-34]. Many researchers have extensively studied the numerical methods of impulsive differential equations. M.I. Berenguer[35] provide a collage-type theorem for impulsive differential equations with inverse boundary conditions. Epshteyn[36, 37] solved the high-order differential equations with interface conditions based on Difference Potentials approach for the variable coefficient. Hossainzadeh[38] applied the Adomian Decomposition Method(ADM) for solving first-order impulsive differential equations. Zhang[39] researched numerical solutions to the first-order impulsive differential equations by collocation methods. Zhang[40] analyzed a class of linear impulsive delay differential equation by asymptotic stability.

In this chapter, we consider the following second-order impulsive differential equations:

$$\begin{cases} u''(x) + a_1(x)u'(x) + a_0(x)u(x) = f(x), \quad x \in (a,b) \setminus \{c\} \\ u(a) = \alpha_1, \ u(b) = \alpha_2 \\ \Delta u'(c) = \alpha_3, \ \Delta u(c) = \alpha_4 \end{cases} \tag{3-1}$$

where $\Delta u'(c) = u'(c^+) - u'(c^-)$, α_3 and α_4 are not at the same time as 0. $a_i(x)$ and $f(x)$ are known function, $\alpha_j \in \mathbb{R}$, $j = 1, 2, 3, 4$. In this chapter, only one pulse point is considered, by that analogy, the algorithm can also be applied to multiple pulse points.

As known to all, the reproducing kernel method is a powerful tool to solve differential equations[41]. However, the reproducing kernel space is smooth, in order to solve the impulsive differential equation, for the first time, we propose a broken reproducing kernel space. This chapter cleverly deals with the traditional reproducing kernel space, connecting two spaces with pulse points as the boundary. Each space is a smooth reproducing kernel space, achieving the advantage of solving pulse differential equations and utilizing the relevant theoretical properties of the original reproducing kernel space. In this chapter, it is assumed that the solution to the Eq. (3-1) exists uniquely and only one pulse point is considered, while the case of multiple pulse points can be treated similarly.

The aim of this chapter is to derive the numerical solutions of Eq. (3-1) in section 1. In section 2, we introduce the reproducing kernel space for solving problems. The numerical algorithm and convergence order of approximate solution is presented in section 3. In the section 4, the presented algorithms are applied to some numerical experiments. Then we end with some conclusions in section 5.

3.2 Discontinuous reproducing kernel method

Referring to the relevant theory of the reproducing kernel space in the middle of Chapter 2, assuming the space $W^3_{2,c}[a,b]$, $W^1_{2,c}[a,b]$ and its inner product are defined by definition 2.2. In order to solve Eq. (3-1), we introduce a linear operator $\mathcal{L} : W^3_{2,c}[a,b] \to W^1_{2,c}[a,b]$, where

$$\mathcal{L}u = u''(x) + a_1(x)u'(x) + a_0(x)u(x), \quad u \in W^3_{2,c}[a,b]$$

By Ref.[7], it's easy to prove that $\mathcal{L}$ is a bounded operator.

Then Eq. (3-1) can be transformed into the following form

$$\mathcal{L}u = f(x), \quad x \in (a, b) \setminus \{c\} \tag{3-2}$$

where,u satisfies the following boundary conditions

$$\begin{cases} u(a) = \alpha_1, u(b) = \alpha_2 \\ \Delta u'(c) = \alpha_3, \Delta u(c) = \alpha_4 \end{cases}$$

We make $\{x_i\}_{i=1}^{\infty}$ is a dense point set that removed the point c on the interval $[a, b]$, put

$$\begin{cases} \phi_1(x) = R_a(x), \quad \phi_2(x) = R_b(x) \\ \phi_3(x) = \dfrac{\partial R_x(t)}{\partial t}|_{t=c^+} - \dfrac{\partial R_x(t)}{\partial t}|_{t=c^-}, \quad \phi_4(x) = R_x(c^+) - R_x(c^-) \end{cases}$$

and

$$\psi_i(x) = \mathcal{L}^* r_{x_i}(x), \quad i = 1, 2, \cdots \tag{3-3}$$

where $\mathcal{L}^*$ is the adjoint operator of $\mathcal{L}$.

Let $S_n = span\{\{\psi_i(x)\}_{i=1}^{n} \bigcup \{\phi_j(x)\}_{j=1}^{4}\}$. Then we can obtain that $S_n \subset W_{2,c}^3[a, b]$. The orthogonal projection operator are denoted by $\mathcal{P}_n : W_{2,c}^3[a, b] \to S_n$.

Theorem 3.1 $\psi_i(x) = \mathcal{L}R_x(x_i), \quad i = 1, 2, \cdots$.

Proof: By Eq. (3-3) of ψ_i and the reproducibility of kernel functions, we have

$$\begin{aligned} \psi_i(x) &= \langle \mathcal{L}^* r_{x_i}, R_x \rangle_{W_{2,c}^3} = \langle r_{x_i}, \mathcal{L}R_x \rangle_{W_{2,c}^1} \\ &= \mathcal{L}R_x(x_i), \ i = 1, 2, \cdots \end{aligned}$$

□

Theorem 3.2 $\{\psi_i(x)\}_{i=1}^{n} \bigcup \{\phi_i(x)\}_{i=1}^{4}$ is linearly independent in $W_{2,c}^3[a, b]$.

Proof: Let $0 = \sum_{i=1}^{n} \lambda_i \psi_i(t) + \sum_{j=1}^{4} k_j \phi_j(t)$. The following proof shows that coefficients $\lambda_i, k_j, i = 1, 2, \cdots, n, j = 1, 2, 3, 4$ are all 0. Consider $h(t) \in W_{2,c}^3[a, b]$, and it satisfies

$$\begin{cases} \mathcal{L}h = 0, \quad t \in (a, b) \setminus \{c\} \\ h(a) = 0, h(b) = 0, \\ \Delta h'(c) = 1, \Delta h(c) = 0 \end{cases}$$

then

$$
\begin{aligned}
0 &= \langle h(t), \sum_{i=1}^{n} \lambda_i \psi_i(t) + \sum_{j=1}^{4} k_j \phi_j(t) \rangle \\
&= \sum_{i=1}^{n} \lambda_i \langle h(t), \mathcal{L}^* r_{x_i}(t) \rangle + k_1 \langle h(t), R_a(t) \rangle + k_2 \langle h(t), R_b(t) \rangle \\
&\quad + k_3 \langle h(t), \frac{\partial R_x(t)}{\partial x}|_{x=c^+} - \frac{\partial R_x(t)}{\partial x}|_{x=c^-} \rangle + k_4 \langle h(t), R_{c^+}(t) - R_{c^-}(t) \rangle \\
&= \sum_{i=1}^{n} \lambda_i \mathcal{L} h(x_i) + k_1 h(a) + k_2 h(b) + k_3 (h'(c^+) - h'(c^-)) + k_4 (h(c^+) - h(c^-)) \\
&= k_3
\end{aligned}
$$

Similarly, we have $k_1 = 0, k_2 = 0, k_4 = 0$.

Consider $f_j(t)\begin{cases} = 0, & t = x_1, x_2, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n \\ \neq 0, & \text{others} \end{cases}$, and $v_j(t) \in W_{2,c}^3[a,b]$ satisfies

$$
\begin{cases} \mathcal{L} v_j = f_j(t), & t \in (a,b) \setminus \{c\} \\ v_j(a) = 0, v_j(b) = 0, \end{cases} \tag{3-4}
$$

The unique solution to the above equations exist(see ref.[7]), then

$$
\begin{aligned}
0 = \langle v_j(t), \sum_{i=1}^{n} \lambda_i \psi_i(t) \rangle &= \sum_{i=1}^{n} \lambda_i \langle v_j(t), \mathcal{L}^* r_{x_i}(t) \rangle \\
&= \sum_{i=1}^{n} \lambda_i \mathcal{L} v_j(x_i) = \sum_{i=1}^{n} \lambda_i f_j(x_i) \\
&= \lambda_j f_j(x_j)
\end{aligned}
$$

So, $\lambda_j = 0, j = 1, 2, \cdots, n$. □

Theorem 3.3 If $u \in W_{2,c}^3[a,b]$ is the solution of Eq. (3-2), then $u_n = \mathcal{P}_n u$ satisfies the following equations

$$
\begin{cases} \langle v, \psi_i \rangle = f(x_i), & i = 1, 2, \cdots, n \\ \langle v, \phi_1 \rangle = \alpha_1, \langle v, \phi_2 \rangle = \alpha_2, \langle v, \phi_3 \rangle = \alpha_3, \langle v, \phi_4 \rangle = \alpha_4 \end{cases} \tag{3-5}
$$

Proof: Supposing $u(x)$ is a solution of Eq. (3-2), then

$$
\begin{aligned}
\langle \mathcal{P}_n u, \psi_i \rangle_{W_{2,c}^3} &= \langle u, \mathcal{P}_n \psi_i \rangle_{W_{2,c}^3} = \langle u, \psi_i \rangle_{W_{2,c}^3} = \langle u, \mathcal{L}^* r_{x_i} \rangle_{W_{2,c}^3} \\
&= \langle \mathcal{L} u, r_{x_i} \rangle_{W_{2,c}^1} = \mathcal{L} u(x_i) = f(x_i)
\end{aligned}
$$

and

$$
\langle \mathcal{P}_n u, \phi_1 \rangle_{W_{2,c}^3} = \langle u, \mathcal{P}_n \phi_1 \rangle_{W_{2,c}^3} = \langle u, \phi_1 \rangle_{W_{2,c}^3} = \langle u, R_a \rangle_{W_{2,c}^3} = u(a) = \alpha_1
$$

Similarly, we have $\langle \mathcal{P}_n u, \phi_2 \rangle = \alpha_2$, $\langle \mathcal{P}_n u, \phi_3 \rangle = \alpha_3$, $\langle \mathcal{P}_n u, \phi_4 \rangle = \alpha_4$.

So, $\mathcal{P}_n u$ is the solution of Eq. (3-5) . □

In fact, $u_n(x)$ is an approximate solution of the exact solution. Through the knowledge theory in this section, the true solution of Eq. (3-2) can be expressed as

$$u(x) = \sum_{i=1}^{\infty} \lambda_i \psi_i(x) + k_1\phi_1(x) + k_2\phi_2(x) + k_3\phi_3(x) + k_4\phi_4(x) \tag{3-6}$$

Based on the previous knowledge analysis, it is known that the solution u_n is the approximate solution of Eq. (3-1) . Considering $u_n \in S_n$, truncate Eq. (3-6) to obtain an approximate solution

$$u_n(x) = \sum_{i=1}^{n} \lambda_i \psi_i(x) + k_1\phi_1(x) + k_2\phi_2(x) + k_3\phi_3(x) + k_4\phi_4(x) \tag{3-7}$$

To obtain the approximate solution u_n, we only need to obtain the coefficients of each $\psi_i(x)(i = 1, 2, \cdots, n)$ and $\phi_j(x)(j = 1, 2, 3, 4)$. Use $\psi_i(x)$ and $\phi_j(x)$ to do the inner products with both sides of Eq. (3-7) , we have

$$\begin{cases} \sum_{j=1}^{n} \lambda_j \langle \psi_j, \psi_i \rangle + \sum_{j=1}^{4} k_j \langle \psi_i, \phi_j \rangle = f(x_i), \quad i = 1, 2, \cdots, n \\ \sum_{j=1}^{n} \lambda_j \langle \psi_j, \phi_i \rangle + \sum_{j=1}^{4} k_j \langle \phi_i, \phi_j \rangle = \alpha_i, \quad i = 1, 2, 3, 4 \end{cases} \tag{3-8}$$

This is the system of linear equations of $\lambda_i, k_j, i = 1, 2, \cdots, n, \quad j = 1, 2, 3, 4$.

Let

$$G_{n+4} = \begin{bmatrix} \langle \psi_i, \psi_k \rangle & \cdots & \langle \psi_i, \phi_j \rangle \\ \cdots & \cdots & \cdots \\ \langle \psi_k, \phi_j \rangle & \cdots & \langle \phi_j, \phi_m \rangle \end{bmatrix}_{i,k=1,2,\cdots,n, \quad j,m=1,2,3,4}$$

$$F = (f(x_1), f(x_2), \cdots, f(x_n), \alpha_1, \alpha_2, \alpha_3, \alpha_4)^{\mathrm{T}}$$

Consider that $\{\psi_i(x)\}_{i=1}^{n} \bigcup \{\phi_j(x)\}_{j=1}^{4}$ is linearly independent in $W_{2,c}^3[a, b]$, so, G^{-1} is exist. Then, we have

$$(\lambda_1, \lambda_2, \cdots, \lambda_n, k_1, k_2, k_3, k_4)^{\mathrm{T}} = G_{n+4}^{-1} F$$

3.3 Convergence order and stability analysis

Theorem 3.4 If $u \in W_{2,c}^3[a, b]$ is the solution of Eq. (3-2) , then u_n uniformly converges to u.

Proof: It can be inferred from the properties related to reproducing kernel

$$\begin{aligned}|u-u_n| &= |\langle u-u_n, \frac{\partial R_t}{\partial t}\rangle| \\ &\le \|\frac{\partial R_t}{\partial t}\|_{W^3_{2,c}} \ \|u-u_n\|_{W^3_{2,c}} \le M\|u-u_n\|_{W^3_{2,c}} \to 0\end{aligned}$$

□

Theorem 3.5 u_n converges to u, and has second-order convergence.

Proof: By definition 2.2 ,we have

$$u(x) = \begin{cases} u_0(x), & x < c \\ u_1(x), & x \ge c \end{cases}$$

where, $u_0(x) \in W^3_a, \ u_1(x) \in W^3_b$,then

$$\|u(x)-u_n(x)\|^2_{W^3_{2,c}} = \|u_0(x)-u_{0,n}(x)\|^2_{W^3_a} + \|u_1(x)-u_{1,n}(x)\|^2_{W^3_b} \to 0$$

So,

$$\|u_0(x)-u_{0,n}(x)\|^2_{W^3_a} \to 0, \ \|u_1(x)-u_{1,n}(x)\|^2_{W^3_b} \to 0$$

Note that,W^3_a and W^3_b are both reproducing kernel spaces,and according to ref.[15], it can be inferred that, $u_{0,n}$ and $u_{1,n}$ converge to u_0 and u_1 respectively, and have second-order convergence. So,

$$\begin{aligned}|u(x)-u_n(x)|^2 &\le M^2\|u(x)-u_n(x)\|^2_{W^3_{2,c}} \\ &= M^2(\|u_0(x)-u_{0,n}(x)\|^2_{W^3_a} + \|u_1(x)-u_{1,n}(x)\|^2_{W^3_b}) \\ &\le M^2((M_0h^2)^2 + (M_1h^2)^2) = M_2^2h^4\end{aligned}$$

So,

$$|u(x)-u_n(x)| \le M_2h^2$$

where,h is the step size, M, M_0, M_1, M_2 are constants. □

Similarly, it can be proven that if $x \in [a,c)$ or $x \in (c,b]$, $u_n^{(i)}$ uniformly converges to $u^{(i)}, i = 1,2$.

Furthermore, the calculation formula for the convergence order can be given as follows

$$\text{C.R} = \log_2 \frac{\max|u(x)-u_n(x)|}{\max|u(x)-u_{2n}(x)|}$$

Theorem 3.6 Set $u \in W^3_{2,c}$ is the true solution of Eq. (3-2) , if there is a slight perturbation δ in the right-hand term f of Eq. (3-2) , denoted as $\widetilde{f} = f + \delta$. And $\widetilde{u}$ satisfies $\mathcal{L}\widetilde{u} = \widetilde{f}$,

then

$$|u - \widetilde{u}| \leq M|\delta|$$

where,M is constant.

Proof: According to the uniqueness of the solution of Eq. (3-2) and the property of reproducing kernel space, it can be known that $\mathcal{L}$ is a bounded linear operator from $W_{2,c}^3$ to $W_{2,c}^1$ and is a one-to-one mapping. Therefore, $\mathcal{L}^{-1}$ exists and is bounded, so

$$\begin{aligned} |u - \widetilde{u}| &= |\mathcal{L}^{-1} f - \mathcal{L}^{-1} \widetilde{f}| = |\mathcal{L}^{-1} f - \mathcal{L}^{-1}(f + \delta)| \\ &= |\mathcal{L}^{-1}\delta| \leq \|\mathcal{L}^{-1}\| \, |\delta| \leq M|\delta| \end{aligned}$$

So,for $\forall\varepsilon > 0, \exists\delta = \frac{\varepsilon}{M}$,we have $|u - \widetilde{u}| < \varepsilon$. From this, it can be seen that the numerical algorithm designed in this chapter is not sensitive to errors within a certain range. Even when there are weak disturbances in the calculation process, the approximate and exact solutions can still maintain a good degree of approximation. □

From the relevant theories in this section, it can be seen that the numerical algorithm designed based on the discontinuous reproducing kernel space framework has the ability to solve pulse problems. In addition, combining with the definition of discontinuous reproducing kernel space, it can be seen that the algorithm does not require high regularity of the solution to the function, and does not need to consider the continuity constraints in general reproducing kernel methods to operate on the function.

3.4 Numerical examples

In this section, the method proposed in this chapter is applied to some impulsive differential equations to evalute the approximate solution. Example 3.1 , both example 3.2 and example 3.3 are calculated using the regenerated kernel space $W_{2,c}^3[0, 1]$, and n is the number of dense point on the interval $[0, 1]$. We compare the numerical results with the other methods dicussed in ref.[35, 36]. Finally, the results show that our algorithm is practial and remarkably effective.

Example 3.1 Consider the linear impulsive differential equation[35]

$$\begin{cases} -u''(x) + u(x) = 0, \quad x \in (0, 1) \setminus \{1/4\} \\ u(0) = 0, \quad u(1) = -1 \\ \Delta u(1/4) = 0, \quad \Delta u'(1/4) = -2 \end{cases}$$

The exact solution of example 3.1 is

$$u(x)=\begin{cases}\dfrac{e^{\frac{1}{4}-x}\left(-1-e^{\frac{3}{4}}+e^{\frac{3}{2}}\right)\left(e^{2x}-1\right)}{e^{2}-1}, & x\in[0,\frac{1}{4}]\\ \dfrac{e^{-\frac{1}{4}-x}\left(e^{2x}-e^{2x+\frac{1}{2}}-e^{2x+\frac{5}{4}}+e^{\frac{5}{4}}-e^{2}+e^{\frac{5}{2}}\right)}{e^{2}-1}, & x\in(\frac{1}{4},1]\end{cases}$$

Table 3-1: Comparison of absolute errors in Example 3.1

n	$\lvert u(x)-u_n(x)\rvert$[35]	our method				
		$\max\lvert u-u_n\rvert$	C.R	$\max\lvert u'-u'_n\rvert$	C.R	$\max\lvert u''-u''_n\rvert$
33	3.426E-3	7.988E-5		3.381E-4		7.988E-5
129	8.562E-4	5.297E-6		2.252E-5		5.297E-6
200	–	4.563E-6		1.896E-5		4.563E-6
400	–	4.486E-7	3.3464	1.934E-6	3.2933	4.486E-7

Table 3-1 presents the numerical calculation results and compares them with the error results in ref.[35]. The numerical results are consistent with the relevant theories of theorem 3.4 . The results indicate that the method proposed in this chapter can obtain more accurate approximate solutions. From the illustrative tables, we conclude that when truncation limit n is increased we can obtain a good accuracy.

Example 3.2 Consider the following equation with two pulse points[35]

$$\begin{cases}-u''(x)=0, \quad x\in(0,1)\setminus\{1/3,4/5\}\\ u(0)=0, \quad u(1)=0\\ \Delta u(1/3)=0, \quad \Delta u'(1/3)=-1\\ \Delta u(4/5)=0, \quad \Delta u'(4/5)=1\end{cases}$$

The exact solution of example 3.2 is

$$u(x)=\begin{cases}\frac{7x}{15} & x\in[0,\frac{1}{3}]\\ \frac{1}{3}-\frac{8x}{15} & x\in(\frac{1}{3},\frac{4}{5}]\\ -\frac{7}{15}-\frac{7x}{15} & x\in(\frac{4}{5},1]\end{cases}$$

In Fig. 3-1 , the red dotted line is the numerical solution and the black line is the exact solution, Fig. 3-2 shows the variation of absolute error between the approximate solution and the true solution when $n=33$. Table 3-2 shows the comparison results of our algorithm with other methods. All figures and tables indicate that our proposed method is effective, it indicates that our presented method is in good agreement with other methods. It's worth noting that the approximate solutions of Example 3.1 and Example 3.2 are

Table 3-2: Comparison of absolute errors in Example 3.2

n	$\|u(x)-u_n(x)\|$[35]	our method			
		$\max\|u(x)-u_n(x)\|$	C.R	$\max\|u'(x)-u'_n(x)\|$	C.R
33	4.627E-2	3.299E-6		1.197E-5	
129	1.578E-5	2.028E-7		8.090E-7	
400	–	2.048E-8		8.238E-8	
800	–	5.022E-9	2.0271	2.028E-8	2.0222

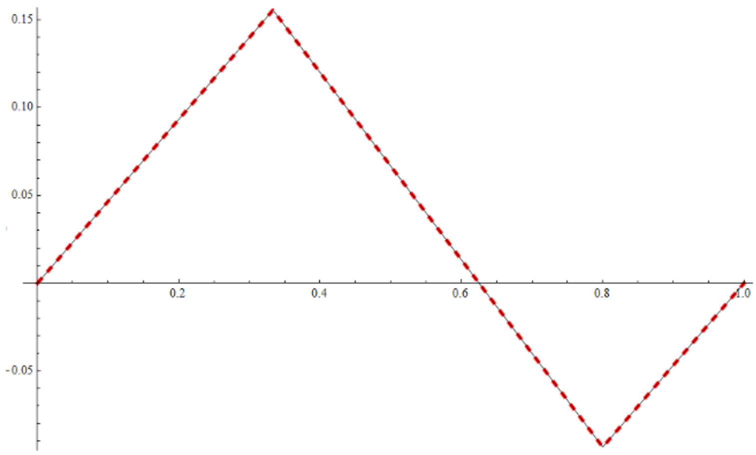

Fig. 3-1: The exact solution and the approximate solution in Example 3-2

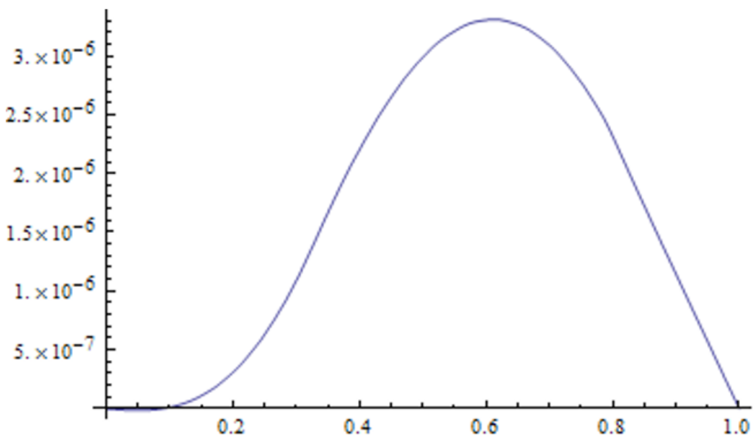

Fig. 3-2: The absolute errors of $|u(x) - u_n(x)|$ in Example 3-2

only proved norm of convergence to the exact solutions(see ref.[35]), but the approximate solutions of this chapter are proved uniformly converges to $u(x)$.

Example 3.3 Consider the following impulsive equation with variable coefficients[36]

$$(\beta u_x)_x = 56x^6, x \in (0,1) \setminus \{0.5\}, \quad \text{where} \quad \beta = \begin{cases} 1, & 0 < x < 0.5 \\ 2, & 0.5 < x < 1 \end{cases}$$

subject to the following boundary and interface conditions

$$\begin{cases} u(0) = 0, & u(1) = \frac{257}{512} \\ \Delta u(0.5) = 0, & \Delta u'(0.5) = -0.5u'(0.5^-) \end{cases}$$

The exact solution of example 3.3 is

$$u(x) = \begin{cases} x^8, & x \in [0, 0.5] \\ \frac{1}{2}\left(x^8 + \frac{1}{256}\right), & x \in (0.5, 1] \end{cases}$$

Table 3-3: Comparison of absolute errors in Example 3.3

n	$\lvert u(x) - u_n(x)\rvert$[36]	our method			
		max $\lvert u(x) - u_n(x)\rvert$	C.R	max $\lvert u'(x) - u'_n(x)\rvert$	C.R
20	1.975E-2	3.221E-3	–	2.849E-2	–
40	6.241E-3	7.207E-4	2.1600	6.832E-3	2.0602
80	1.743E-3	1.702E-4	2.0822	1.669E-3	2.0333
160	4.600E-4	4.132E-5	2.0423	4.123E-4	2.0172
320	1.181E-4	1.017E-5	2.0255	1.024E-4	2.0095
640	2.992E-5	2.524E-6	2.0105	2.553E-5	2.0040

Table 3-4: The absolute errors of added the disturbance in Example 3.3

n	max $\lvert u(x) - u_n(x)\rvert$	max $\lvert u'(x) - u'_n(x)\rvert$
20	3.228E-3	2.852E-2
40	7.273E-4	6.863E-3
80	1.765E-4	1.700E-3
160	4.766E-5	4.436E-4
320	1.701E-5	1.337E-4
640	1.077E-5	5.678E-5

Table 3-3 lists the absolute error and convergence order of example 3.3 . The data in the table indicates that when truncating nnincreases, higher accuracy approximate solutions can be obtained, indicating that the proposed method is very stable and effective. This method can not only solve the Eq. (3-1) of pulse differential equations, but also be extended to solve high-order pulse differential equations and complex boundary value problems of pulses. Table 3-4 shows the result of adding the disturbance 10^{-4} on the right-hand side $f(x)$, it indicates that disturbance is hardly affect the results of our method. It shows that the proposed approach is very stable and effective.

Example 3.4 Consider the three-order linear impulsive differential equation

$$\begin{cases} u'''(x) + a_2(x)u''(x) + a_0(x)u(x) = f(x), & x \in (-2, 2) \setminus \{0\} \\ u(-2) = 16, \ \ u(2) = \frac{14}{3}, \ \ \int_{-1}^{1} u(x)dx = \frac{119}{120} \\ \Delta u(0) = 0, \ \ \Delta u'(0) = 1, \ \ \Delta u''(0) = 2 \end{cases}$$

where, $a_0(x) = \begin{cases} 1 - x, & x < 0 \\ e^{-x}, & x \geq 0 \end{cases}$ $\quad a_2(x) = \begin{cases} -2\cos(x), & x < 0 \\ -2, & x \geq 0 \end{cases}$

$$f(x) = \begin{cases} x\left(-x^4 + x^3 - 24x\cos(x) + 24\right), & x < 0 \\ e^{-x}\left(-\frac{x^3}{6} + x^2 + x\right) + 2(x-2) - 1, & x \geq 0 \end{cases}$$

The exact solution of example 3.4 is

$$u(x) = \begin{cases} x^4, & x < 0 \\ -\frac{x^3}{6} + x^2 + x, & x \geq 0 \end{cases}$$

Table 3-5: Comparison of absolute errors in Example 3.4

n	$\max \lvert u - u_n \rvert$	C.R	$\max \lvert u' - u_n' \rvert$	$\max \lvert u'' - u_n'' \rvert$	$\max \lvert u''' - u_n''' \rvert$
320	1.8367E-5	–	7.0934E-5	1.9598E-4	3.9198E-4
640	4.6155E-6	2	1.7919E-5	4.9411E-5	9.8826E-5

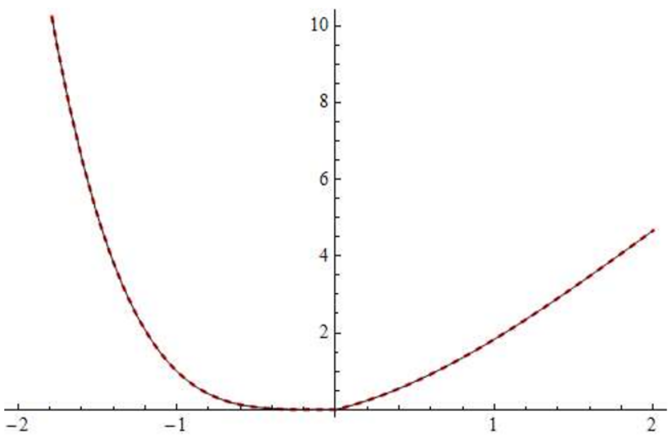

Fig. 3-3: The approximate solution u_n and the exact solution u in Example 3.4

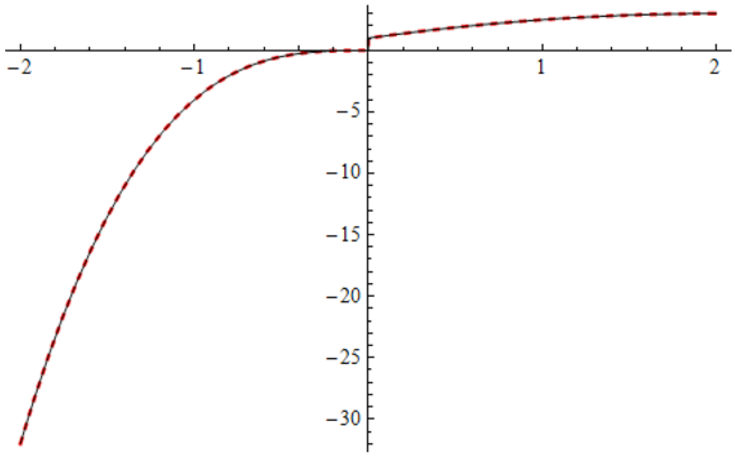

Fig. 3-4: The approximate solution u'_n and the exact solution u' in Example 3.4

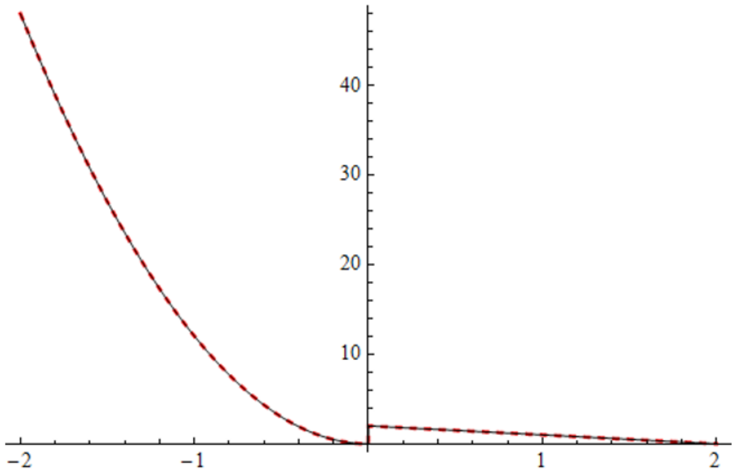

Fig. 3-5: The approximate solution u''_n and the exact solution u'' in Example 3.4

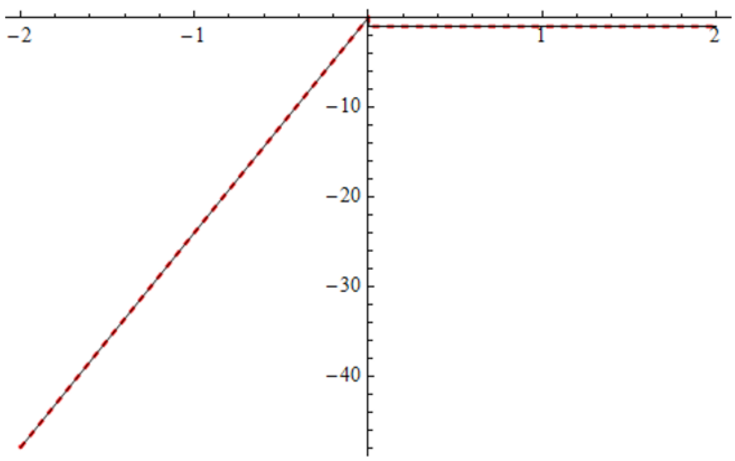

Fig. 3-6: The approximate solution u'''_n and the exact solution u''' in Example 3.4

In example 3.4 , the coefficients $a_0(x), a_2(x)$ also have the characteristic of piecewise smoothness, n is the number of dense point on the interval $[-2, 2]$, the reproducing space is $W^4_{2,c}[-2, 2]$. Fig. 3-3 show the approximation between approximate solutions and true solutions, table 3-5 and Fig. 3-4 , Fig. 3-5 , and Fig. 3-6 respectively show the approximation between the derivatives of approximate solutions and true solutions, and the variation law conforms to the theorem 3.4 . Among them, the red dashed line represents approximate solution $u_n^{(i)}(x)$, and the black curve represents true solution $u^{(i)}(x),\ i = 0, 1, 2, 3$. The results shows that the present method is also effective for high-order impulsive differential equations and complex boundary value problems of pulse.

3.5 Conclusion

This chapter applies the simplified reproducing kernel method to the solution of pulse differential equations, cleverly establishing a discontinuous reproducing kernel s-pace with piecewise smooth properties, and transforming the differential equation to be solved into an equivalent operator equation in this space to design an algorithm for solving. We have designed numerical algorithms for solving second-order pulse differential equations, delayed pulse differential equations, and piecewise constant pulse differential equations, with a focus on describing the construction process of the discontinuous regeneration accounting method. We have comprehensively utilized linear multi-step method and optimization method to design algorithms, and each algorithm has proven relevant theories such as convergence and convergence order. From all the tables and graphs of the examples, it can be seen that, The algorithm designed in this chapter is very accurate and effective.

In fact, this intermittent regeneration kernel idea can be extended to other types of pulse boundary value problems. The proposal of the intermittent regeneration kernel idea is a significant supplement to the traditional regeneration kernel idea, allowing the regeneration kernel method to be applied to solve the problem of shard smoothness. It not only retains the original advantages of the regeneration kernel method, but also expands the application scope of the regeneration kernel theory.

Chapter 4 The reproducing kernel method for nonlinear impulsive differential equations

In this chapter, a new algorithm is presented to solve the nonlinear impulsive differential equations. In the first time, this chapter combines the reproducing kernel method with the least squares method to solve the second-order nonlinear impulsive differential equations. Then the uniform convergence of the numerical solution is proved, and the time consuming Schmidt orthogonalization process is avoided. The algorithm is employed successfully on some numerical examples.

4.1 Second-order nonlinear impulsive differential equation

In recent years, the impulsive differential equation model has been applied to many aspects of life: population dynamics[26], physics, chemistry[27], irregular geometries and interface problems[28-30], signal processing[31]. Many scholars have studied the existence and numerical solution of the impulsive differential equations[32-34]. Epshteyn[36, 37] solved the high-order linear differential equations with interface conditions based on Difference Potentials approach for the variable coefficient. However, so far, no scholars have discussed the numerical solution of the second-order nonlinear impulsive differential equations. Only a few scholars have studied the existence of solutions[42]. Sadollaha[43] suggested a least square algorithm to solve a wide variety of linear and nonlinear ordinary differential equations. Zhang[44] presented the reproducing kernel method and least square to nonlinear boundary value problems. These research work shows that the least square method plays a very good role in solving nonlinear problems. As known to all, the reproducing kernel method is a powerful tool to solve differential equations[7, 41, 44, 45]. Al-Smadi[46] introduced a iterative reproducing kernel method and other methods for providing numerical approximate solutions of time-fractional boundary value problem.

In this chapter, we consider the following second-order nonlinear impulsive differential equations(NIDEs for short):

$$\begin{cases} u''(x) + a_1(x)u'(x) + a_0(x)u(x) + N(u) = f(x), & x \in [a,b] \setminus \{c\} \\ u(a) = \alpha_1, u(b) = \alpha_2, \Delta u'(c) = \alpha_3, \Delta u(c) = \alpha_4 \end{cases} \tag{4-1}$$

where $\Delta u'(c) = u'(c^+) - u'(c^-)$, α_3 and α_4 are not at the same time as 0. $a_i(x)$ and $f(x)$ are known function, $\mathcal{N} : \mathbb{R} \to \mathbb{R}$ is a continuous function, $\alpha_j \in \mathbb{R}$, $j = 1, 2, 3, 4$. In this chapter, only one pulse point is considered, by that analogy, the algorithm can also be applied to multiple pulse points.

The aim of this chapter is to derive the numerical solutions of Eq. (4-1) in section 1. In section 2, we introduce the reproducing kernel space for solving problems. The reproducing kernel method and the least squares method are presented in section 3. In the section 4, the presented algorithms are applied to some numerical experiments. Then we end with some conclusions in section 5.

4.2 The broken reproducing kernel space

In this chapter, the traditional reproducing kernel space is dealt with delicately, the space has been broken into two spaces that each one is smooth reproducing kernel space, so we can use this space to solve NIDEs. We assume that the Eq. (4-1) has a unique solution.

4.2.1 The traditional reproducing kernel space

- Reproducing kerenl space $W_2^3[a,c]$ is defined as

$W_2^3[a,c] = \{u(x) | u''$ is an absolutely continuous real value function, $u''' \in L^2[a,c]\}$[20](W_a^3 for short). The inner product and norm are given by ref.[7].

- Reproducing kerenl space $W_2^1[a,c]$ is defined as

$W_2^1[a,c] = \{u(x) | u$ is an absolutely continuous real value function, $u' \in L^2[a,c]\}$[20](W_a^1 for short). The inner product and norm are given by ref.[7].

The reproducing kernel spaces are W_a^3 and W_a^1 with reproducing kernel $R_t^0(x)$ and $r_t^0(x)$, respectively.

In the same way, the reproducing kernel spaces are $W_2^3[c,b]$(W_b^3 for short) and $W_2^1[c,b]$(W_b^1 for short) with reproducing kernel $R_t^1(x)$ and $r_t^1(x)$, respectively.

4.2.2 The reproducing kernel space with piecewise smooth

In this chapter, consider that the exact solution of Eq. (4-1) is not a smooth function, so, we connected two reproducing kernel spaces on both sides of the impulsive point, we call it the broken reproducing kernel space. We have proposed this method for the first time.

Definition 4.1 The linear space $W^3_{2,c}$ is defined as

$W^3_{2,c}[a,b] = \{u(x)|$ if $x < c$ then $u(x) \in W^3_a$, if $x \geq c$ then $u(x) \in W^3_b\}$. Each $u(x) \in W^3_{2,c}[a,b]$ has the following form

$$u(x) = \begin{cases} u_0(x), & x < c \\ u_1(x), & x \geq c \end{cases}$$

where $u_0(x) \in W^3_a, u_1(x) \in W^3_b$.

Theorem 4.1 Assuming that the inner product and norm in $W^3_{2,c}[a,b]$ are given by

$$\langle u, v\rangle_{W^3_{2,c}} = \langle u_0, v_0\rangle_{W^3_a} + \langle u_1, v_1\rangle_{W^3_b}, \qquad u, v \in W^3_{2,c}[a,b] \tag{4-2}$$

$$||u||_{W^3_{2,c}} = \sqrt{\langle u, u\rangle_{W^3_{2,c}}}, \qquad u \in W^3_{2,c}[a,b]$$

then the space $W^3_{2,c}[a,b]$ is a inner space.

Proof: For any $u, v \in W^3_{2,c}[a,b]$,

$$\begin{aligned}\langle u+v, w\rangle_{W^3_{2,c}} &= \langle u_0+v_0, w_0\rangle_{W^3_a} + \langle u_1+v_1, w_1\rangle_{W^3_b} \\ &= \langle u_0, w_0\rangle_{W^3_a} + \langle v_0, w_0\rangle_{W^3_a} + \langle u_1, w_1\rangle_{W^3_b} + \langle v_1, w_1\rangle_{W^3_b} \\ &= \langle u, w\rangle_{W^3_{2,c}} + \langle v, w\rangle_{W^3_{2,c}}.\end{aligned}$$

We can prove that the Eq. (4-2) satisfies the other requirements of the inner product space. □

Theorem 4.2 The space $W^3_{2,c}[a,b]$ is a Hilbert space.

Proof: Suppose that $\{u_n(x)\}_{n=1}^{\infty}$ is a Cauchy sequence in $W^3_{2,c}[a,b]$, however,

$$u_n(x) = \begin{cases} u_{0,n}(x), & x < c \\ u_{1,n}(x), & x \geq c \end{cases} \quad n = 1, 2, \cdots.$$

where $\{u_{0,n}(x)\}_{n=1}^{\infty}$ and $\{u_{1,n}(x)\}_{n=1}^{\infty}$ are Cauchy sequences in W^3_a and W^3_b, respectively.

So, there are two functions $g_0(x) \in W^3_a$, $g_1(x) \in W^3_b$, and

$$||u_{0,n}(x) - g_0(x)||^2_{W^3_a} \to 0, \qquad ||u_{1,n}(x) - g_1(x)||^2_{W^3_b} \to 0.$$

Let

$$g(x) = \begin{cases} g_0(x), & x < c \\ g_1(x), & x \geq c \end{cases}$$

By the definition 4.1 , $g(x) \in W^3_{2,c}[a,b]$, and

$$||u_n(x) - g(x)||^2_{W^3_{2,c}} = ||u_{0,n}(x) - g_0(x)||^2_{W^3_a} + ||u_{1,n}(x) - g_1(x)||^2_{W^3_b} \to 0.$$

So, the space $W^3_{2,c}[a,b]$ is a Hilbert space. □

Theorem 4.3 The space $W^3_{2,c}[a,b]$ is a reproducing kernel space with the reproducing kernel function

$$R_t(x) = \begin{cases} R^0_t(x), & (x,t) \in [a,c) \times [a,c) \\ R^1_t(x), & (x,t) \in [c,b] \times [c,b] \\ 0, & others. \end{cases} \tag{4-3}$$

Proof: Consider arbitrary $u(x) \in W^3_{2,c}[a,b]$.

If $t \in [a,c)$, $\langle u(x), R_t(x)\rangle_{W^3_{2,c}} = \langle u_0(x), R^0_t(x)\rangle_{W^3_a} + \langle u_1(x), 0\rangle_{W^3_b} = u_0(t)$

If $t \in [c,b]$, $\langle u(x), R_t(x)\rangle_{W^3_{2,c}} = \langle u_0(x), 0\rangle_{W^3_a} + \langle u_1(x), R^1_t(x)\rangle_{W^3_b} = u_1(t)$

In conclusions, for every $u(x) \in W^3_{2,c}[a,b]$, it follows that

$$\langle u(x), R_t(x)\rangle = u(t).$$ □

Similarly, the reproducing kernel space $W^1_{2,c}[a,b]$ is defined as

$$W^1_{2,c}[a,b] = \{u(x) | \text{ if } x \langle c \text{ then } u(x) \in W^1_a, \text{ if } x \geq c \text{ then } u(x) \in W^1_b\}, \tag{4-4}$$

and it has the reproducing kernel function

$$r_t(x) = \begin{cases} r^0_t(x), & (x,t) \in [a,c) \times [a,c) \\ r^1_t(x), & (x,t) \in [c,b] \times [c,b] \\ 0, & others. \end{cases} \tag{4-5}$$

In order to solve Eq. (4-1) , we introduce a linear operator $\mathcal{L} : W^3_{2,c}[a,b] \to W^1_{2,c}[a,b]$,

$$\mathcal{L}u = u''(x) + a_1(x)u'(x) + a_0(x)u(x), \quad u \in W^3_{2,c}[a,b]$$

Theorem 4.4 $\mathcal{L}$ is a bounded operator.

Proof: For each fixed $u(x) \in W^3_{2,c}[a,b]$, by difinition 4.1 , $u(x)$ has the following form

$$u(x) = \begin{cases} u_0(x), & x < c \\ u_1(x), & x \geq c \end{cases}$$

where $u_0(x) \in W^3_a, u_1(x) \in W^3_b$.

Moreover,

$$||\mathcal{L}u||^2_{W^1_{2,c}} = ||\mathcal{L}u_0||^2_{W^1_a} + ||\mathcal{L}u_1||^2_{W^1_b}$$

and

$$||\mathcal{L}u_0||^2_{W^1_a} = \langle u_0''(x) + a_1(x)u_0'(x) + a_0(x)u(x), u_0''(x) + a_1(x)u_0'(x) + a_0(x)u(x)\rangle_{W^1_a}$$

$$= [u_0''(a) + a_1(a)u_0'(a) + a_0(a)u(a)]^2 + \int_a^c [(u_0''(x) + a_1(x)u_0'(x) + a_0(x)u(x))']^2 dx$$

$$\leqslant M_1 + \int_a^c (|u_0'''|^2 + M_2|u_0''|^2 + M_3|u_0'|^2 + M_4|u_0|^2 + M_5|u_0||u_0'| + M_6|u_0||u_0''|$$

$$+M_7|u_0||u_0'''| + M_8|u_0'||u_0''| + M_9|u_0'||u_0'''| + M_{10}|u_0''||u_0'''|)dx$$

where $M_i(1 \leqslant i \leqslant 10)$ are constants.

Furthermore,

$$\int_a^c |u^{(i)}||u^{(j)}|dx \leqslant \sqrt{\int_a^c |u^{(i)}|^2 dx \int_a^c |u^{(j)}|^2 dx}$$

and

$$||u_0||^2_{W_a^3} = \langle u_0, u_0 \rangle_{W_a^3} = u_0(a) + u_0'(a) + u_0''(a) + \int_a^c |u_0'''|^2 dx$$

So

$$\int_a^c |u_0^{(i)}||u_0^{(j)}|dx \leqslant C_{ij}||u_0||^2_{W_a^3}, \qquad i, j = 0, 1, 2, 3$$

Therefore

$$||\mathcal{L}u_0||^2_{W_a^1} \leqslant M||u_0||^2_{W_a^3}$$

Where, M and $C_{ij}(i, j = 0, 1, 2, 3)$ are constants.

Similarly, $||\mathcal{L}u_1||^2_{W_b^1} \leqslant M||u_1||^2_{W_b^3}$.

as a result

$$||\mathcal{L}u||^2_{W_{2,c}^1} = ||\mathcal{L}u_0||^2_{W_a^1} + ||\mathcal{L}u_1||^2_{W_b^1} \leqslant M||u||^2_{W_{2,c}^3}$$

In other words, $\mathcal{L}$ is a bounded opertor. □

Then Eq. (4-1) can be transformed into the following form:

$$\begin{cases} \mathcal{L}u = f(x) - \mathcal{N}(u(x)), & x \in [a, b] \setminus \{c\} \\ u(a) = \alpha_1, u(b) = \alpha_2, \Delta u'(c) = \alpha_3, \Delta u(c) = \alpha_4. \end{cases} \tag{4-6}$$

We make $\{x_i\}_{i=1}^{\infty}$ is a dense point set that removed the point c on the interval $[a, b]$, put

$$\phi_1(x) = R_a(x), \phi_2(x) = R_b(x), \phi_3(x) = \frac{\partial R_x(t)}{\partial t}|_{t=c^+} - \frac{\partial R_x(t)}{\partial t}|_{t=c^-}, \phi_4(x) = R_x(c^+) - R_x(c^-)$$

and

$$\psi_i(x) = \mathcal{L}^* r_{x_i}(x). \quad i = 1, 2, \cdots$$

where $\mathcal{L}^*$ is the adjoint operator of $\mathcal{L}$.

Let $S_n = span\{\{\psi_i(x)\}_{i=1}^n \bigcup \{\phi_j(x)\}_{j=1}^4\}$. Then we can obtain that $S_n \subset W_{2,c}^3[a, b]$.

The orthogonal projection operator is denoted by $\mathcal{P}_n : W^3_{2,c}[a,b] \to S_n$, let $u_n \triangleq \mathcal{P}_n u$.

Theorem 4.5 $\psi_i(x) = \mathcal{L}R_x(x_i). \quad i = 1, 2, \cdots$.

Proof: $\psi_i(x) = \langle \mathcal{L}^* r_{x_i}, R_x \rangle_{W^3_{2,c}} = \langle r_{x_i}, \mathcal{L}R_x \rangle_{W^1_{2,c}} = \mathcal{L}R_x(x_i).\ i = 1, 2, \cdots$. □

Theorem 4.6 For each fixed n, $\{\psi_i(x)\}_{i=1}^n \bigcup \{\phi_i(x)\}_{i=1}^4$ is linearly independent in $W^3_{2,c}[a,b]$.

Proof: Let $0 = \sum_{i=1}^{n} \lambda_i \psi_i(t) + \sum_{j=1}^{4} k_j \phi_j(t)$,

- Consider $h(t) \in W^3_{2,c}[a,b]$, $\begin{cases} \mathcal{L}h = 0, & t \in [a,b] \setminus \{c\} \\ h(a) = 0, h(b) = 0, \Delta h'(c) = 1, \Delta h(c) = 0 \end{cases}$, then

$$0 = \langle h(t), \sum_{i=1}^{n} \lambda_i \psi_i(t) + \sum_{j=1}^{4} k_j \phi_j(t) \rangle$$

$$= \sum_{i=1}^{n} \lambda_i \langle h(t), \mathcal{L}^* r_{x_i}(t) \rangle + k_1 \langle h(t), R_a(t) \rangle + k_2 \langle h(t), R_b(t) \rangle$$

$$+ k_3 \langle h(t), \frac{\partial R_x(t)}{\partial x}|_{x=c^+} - \frac{\partial R_x(t)}{\partial x}|_{x=c^-} \rangle + k_4 \langle h(t), R_{c^+}(t) - R_{c^-}(t) \rangle$$

$$= \sum_{i=1}^{n} \lambda_i \mathcal{L}h(x_i) + k_1 h(a) + k_2 h(b) + k_3 (h'(c^+) - h'(c^-)) + k_4 (h(c^+) - h(c^-)) = k_3.$$

Similarly, we have $k_1 = 0, k_2 = 0, k_4 = 0$.

- Consider $f_j(t) \begin{cases} = 0, & t = x_1, x_2, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n \\ \neq 0, & others \end{cases}$, $f_j(t) \in W^1_{2,c}[a,b]$,

take $v_j(t) \in W^3_{2,c}[a,b]$, $\begin{cases} \mathcal{L}v_j = f_j(t), & t \in [a,b] \setminus \{c\} \\ v_j(a) = 0, v_j(b) = 0, \end{cases}$. The unique solution to the above equations exist(see ref.[7]), then

$$0 = \langle v_j, \sum_{i=1}^{n} \lambda_i \psi_i \rangle = \sum_{i=1}^{n} \lambda_i \langle v_j, \mathcal{L}^* r_{x_i} \rangle = \sum_{i=1}^{n} \lambda_i \mathcal{L} v_j(x_i) = \sum_{i=1}^{n} \lambda_i f_j(x_i) = \lambda_j f_j(x_j).$$

So, $\lambda_j = 0, j = 1, 2, \cdots, n$. □

4.3 Primary result

In this section, based on the least square method, the approximate solution of Eq. (4-6) is presented in the broken reproducing kernel space $W^3_{2,c}[a,b]$. And the convergence of the approximate solution is proved.

Theorem 4.7 If $u \in W^3_{2,c}[a,b]$ is the solution of Eq. (4-6) , then u_n satisfies the following:

$$\begin{cases} \langle v, \psi_i \rangle = f(x_i) - \mathcal{N}(u(x_i)), & i = 1, 2, \cdots, n \\ \langle v, \phi_1 \rangle = \alpha_1, \langle v, \phi_2 \rangle = \alpha_2, \langle v, \phi_3 \rangle = \alpha_3, \langle v, \phi_4 \rangle = \alpha_4. \end{cases} \tag{4-7}$$

Proof: Assume u is a solution of Eq. (4-6) , then

$$\begin{aligned}\langle u_n, \psi_i\rangle_{W^3_{2,c}} &= \langle \mathcal{P}_n u, \psi_i\rangle_{W^3_{2,c}} = \langle u, \mathcal{P}_n \psi_i\rangle_{W^3_{2,c}} = \langle u, \psi_i\rangle_{W^3_{2,c}} \\ &= \langle u, \mathcal{L}^* r_{x_i}\rangle_{W^3_{2,c}} = \langle \mathcal{L}u, r_{x_i}\rangle_{W^1_{2,c}} = \mathcal{L}u(x_i) = f(x_i) - \mathcal{N}(u(x_i)).\end{aligned}$$

and

$$\begin{aligned}\langle u_n, \psi_i\rangle_{W^3_{2,c}} &= \langle \mathcal{P}_n u, \phi_1\rangle_{W^3_{2,c}} = \langle u, \mathcal{P}_n \phi_1\rangle_{W^3_{2,c}} = \langle u, \phi_1\rangle_{W^3_{2,c}} \\ &= \langle u, R_a\rangle_{W^3_{2,c}} = u(a) = \alpha_1.\end{aligned}$$

Similarly, we have $\langle u_n, \phi_2\rangle = \alpha_2, \langle u_n, \phi_3\rangle = \alpha_3, \langle u_n, \phi_4\rangle = \alpha_4$. □

In fact, u_n converges uniformly to u in $W^3_{2,c}$.

Theorem 4.8 If $u \in W^3_{2,c}[a,b]$ is the solution of Eq. (4-6) , then u_n uniformly converges to u.

Proof:
$$\begin{aligned}|u(t) - u_n(t)| &= |\langle u - u_n, R_t\rangle| \le \|R_t\|_{W^3_{2,c}} \|u - u_n\|_{W^3_{2,c}} \\ &\le M\|u - u_n\|_{W^3_{2,c}} \to 0.\end{aligned}$$
□

Similarly, we can prove that if $t \in [a,c]$ and $[c,b]$ respectively, then $u_n^{(i)}$ uniformly converges to $u^{(i)}$, $i = 1, 2$.

Since $\mathcal{N}$ is continuous, and $u_n \to u$ uniformly, we have

$$\mathcal{N}(u(x_i)) = \mathcal{N}(\lim_{n\to\infty} u_n(x_i)) = \lim_{n\to\infty} \mathcal{N}(u_n(x_i)).$$

Therefore, while u is the solution of Eq.(6), $u_n = \mathcal{P}_n u$, we have

$$\begin{cases} \langle u_n, \psi_i\rangle = f(x_i) - \mathcal{N}(\lim_{n\to\infty} u_n(x_i)), \quad i = 1, 2, \cdots, n \\ \langle u_n, \phi_1\rangle = \alpha_1, \langle u_n, \phi_2\rangle = \alpha_2, \langle u_n, \phi_3\rangle = \alpha_3, \langle u_n, \phi_4\rangle = \alpha_4. \end{cases} \tag{4-8}$$

So, the approximate solution u_n of Eq. (4-6) is the solution of the following equation.

$$\begin{cases} \langle v, \psi_i\rangle = f(x_i) - \mathcal{N}(v(x_i)) - \varepsilon_i, \quad i = 1, 2, \cdots, n \\ \langle v, \phi_1\rangle = \alpha_1, \langle v, \phi_2\rangle = \alpha_2, \langle v, \phi_3\rangle = \alpha_3, \langle v, \phi_4\rangle = \alpha_4. \end{cases} \tag{4-9}$$

where $\varepsilon_i = \mathcal{N}(u(x_i)) - \mathcal{N}(u_n(x_i)) \to 0$ if $n \to \infty$.

As $u_n \in S_n$, so

$$u_n(x) = \sum_{i=1}^{n} \lambda_i \psi_i(x) + k_1\phi_1(x) + k_2\phi_2(x) + k_3\phi_3(x) + k_4\phi_4(x) \tag{4-10}$$

To obtain the approximate solution u_n, we only need to obtain the coefficients of each $\psi_i(x) (i = 1, 2, \cdots, n)$ and $\phi_j(x) (j = 1, 2, 3, 4)$. Use $\psi_i(x)$ and $\phi_j(x)$ to do the inner products with both sides of Eq. (4-10) , we have

$$\begin{cases} \sum_{j=1}^{n} \lambda_j \langle \psi_j, \psi_i \rangle + \sum_{j=1}^{4} k_j \langle \psi_i, \phi_j \rangle = f(x_i) - \mathcal{N}(u_n(x_i)) - \varepsilon_i, \quad i = 1, 2, \cdots, n \\ \sum_{j=1}^{n} \lambda_j \langle \psi_j, \phi_i \rangle + \sum_{j=1}^{4} k_j \langle \phi_i, \phi_j \rangle = \alpha_i, \quad i = 1, 2, 3, 4 \end{cases} \tag{4-11}$$

This is the system of linear equations of $\lambda_i, k_j, i = 1, 2, \cdots, n, \quad j = 1, 2, 3, 4$.

Let

$$G_{n+4} = \begin{bmatrix} \langle \psi_i, \psi_k \rangle & \cdots & \langle \psi_i, \phi_j \rangle \\ \cdots & \cdots & \cdots \\ \langle \psi_k, \phi_j \rangle & \cdots & \langle \phi_j, \phi_m \rangle \end{bmatrix}_{i,k=1,2,\cdots,n, \quad j,m=1,2,3,4}$$

$\mathcal{N}(u_n(x_i)) + \varepsilon_i = \eta_i, \quad i = 1, 2, \cdots, n.$

$F = (f(x_1) - \eta_1, f(x_2) - \eta_2, \cdots, f(x_n) - \eta_n, \alpha_1, \alpha_2, \alpha_3, \alpha_4)^T.$

Consider that $\{\psi_i(x)\}_{i=1}^n \bigcup \{\phi_j(x)\}_{j=1}^4$ is linearly independent in $W_{2,c}^3[a, b]$, so, G^{-1} is exist. Then, we have

$$(\lambda_1, \lambda_2, \cdots, \lambda_n, k_1, k_2, k_3, k_4)^T = G^{-1} \cdot F. \tag{4-12}$$

So, $\lambda_1, \lambda_2, \cdots, \lambda_n, k_1, k_2, k_3, k_4$ are expressed by $\eta_1, \eta_2, \cdots, \eta_n$.

Substituting Eq. (4-12) into Eq. (4-10) yields

$$u_n(x) = \sum_{i=1}^{n} \lambda_i(\eta_1, \eta_2, \cdots, \eta_n)\psi_i(x) + \sum_{j=1}^{4} k_j(\eta_1, \eta_2, \cdots, \eta_n)\phi_j(x). \tag{4-13}$$

In order to solve the approximate solution u_n of Eq. (4-6) , it is necessary to make $\mathcal{N}(u(x_i))$ and $\mathcal{N}(u_n(x_i))$ close to the maximum, that is to say, each ε_i is as close as possible to 0. Therefore, we construct the following optimization model to solve the value of $(\eta_1, \eta_2, \cdots, \eta_n)$

$$\min \sum_{i=1}^{n} (\mathcal{N}(u_n(x_i)) - \eta_i)^2. \tag{4-14}$$

For the above model, it is actually a common nonlinear optimization problem, and there are many mature methods to solve the problem. In this chapter, the least square method is used to solve the minimum point $(\eta_1^0, \eta_2^0, \cdots, \eta_n^0)$ of Eq. (4-14) , and Mathematica software is used to implement the program, substituting $(\eta_1^0, \eta_2^0, \cdots, \eta_n^0)$ into Eq. (4-13) yields the solution u_n of Eq. (4-10) , namely, u_n is the approximate solution of Eq. (4-6) .

4.4 Numerical examples

In this section, the method proposed in this chapter is applied to some impulsive differential equations to evaluate the approximate solution, and the reproducing space is $W_{2,c}^3[0,1]$. Finally, the results show that our algorithm is practical and remarkably effective.

Example 4.1 Consider the nonlinear impulsive differential equation

$$\begin{cases} u''(x) + xu'(x) + u(x) + u(x)u'(x) = f(x), \quad x \in [0,1] \setminus \{\frac{1}{2}\} \\ u(0) = 0, \ \ u(1) = 2, \ \ \Delta u(\frac{1}{2}) = \frac{11}{16}, \ \ \Delta u'(\frac{1}{2}) = \frac{3}{2}. \end{cases}$$

where

$$f(x) = \begin{cases} x^2\left(4x^5 + 5x^2 + 12\right) \ x \in [0, \frac{1}{2}) \\ 2x^3 + 6x^2 + 3x + 2 \ \ x \in [\frac{1}{2}, 1] \end{cases}$$

The exact solution

$$u(x) = \begin{cases} x^4 \qquad x \in [0, \frac{1}{2}) \\ x^2 + x \ x \in [\frac{1}{2}, 1] \end{cases}$$

Table 4-1: Comparison of absolute errors in Example 4.1 ($n = 32$)

x	True solution	Approximate solution	Absolute error
0	0	0	0
0.2	0.0016	0.0022004090268169	6.0041E-4
0.4	0.0256	0.0268323189078403	1.2323E-3
0.6	0.96	0.9611287501607335	1.1288E-3
0.8	1.44	1.4404744918697479	4.7449E-4
1	2	1.9999999999998206	1.7941E-13

Example 4.2 Consider the nonlinear impulsive differential equation

$$\begin{cases} u''(x) - u(x) + u^3(x) = f(x), \quad x \in [0,1] \setminus \{\frac{1}{2}\} \\ u(0) = 0, \ \ u(1) = \frac{e-1}{2e^{3/2}}, \ \ \Delta u(\frac{1}{2}) = 0, \ \ \Delta u'(\frac{1}{2}) = -1. \end{cases}$$

where

$$f(x) = \begin{cases} \frac{1}{8}e^{-3x-\frac{3}{2}}\left(e^{2x} - 1\right)^3 \ x \in [0, \frac{1}{2}) \\ \frac{1}{8}(e-1)^3 e^{-3x-\frac{3}{2}} \qquad x \in [\frac{1}{2}, 1] \end{cases}$$

The exact solution

$$u(x) = \begin{cases} \frac{1}{2}e^{-\frac{1}{2}}\left(e^x - e^{-x}\right) \ x \in [0, \frac{1}{2}) \\ \frac{1}{2}(e^{\frac{1}{2}} - e^{-\frac{1}{2}})e^{-x} \ x \in [\frac{1}{2}, 1] \end{cases}$$

Table 4-2: Comparison of absolute errors in Example 4.2 ($n = 32$)

x	True solution	Approximate solution	Absolute error
0	0	2.8865798640254E-15	2.8866E-15
0.2	0.12211645844515	0.12216608333390094	4.9625E-5
0.4	0.24913387914768	0.24924050619878646	1.0663E-4
0.6	0.28598316716894	0.28609078503050855	1.0762E-4
0.8	0.23414321382385	0.23419550403227257	5.2290E-5
1	0.19170024978210	0.19170024978210173	2.7755E-17

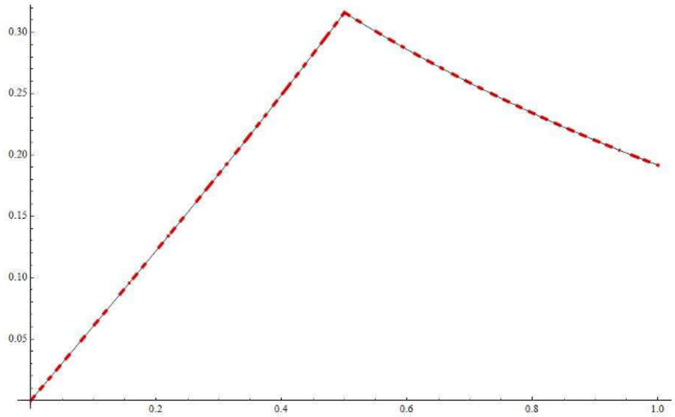

Fig. 4-1: The approximate solution u_n and the exact solution u in Example 4.2 ($n = 32$)

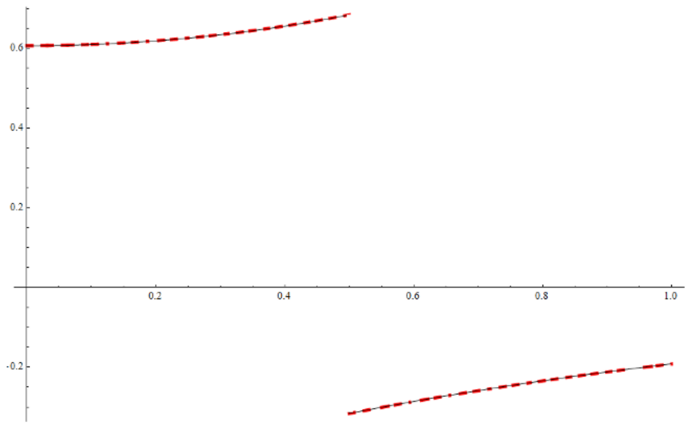

Fig. 4-2: The approximate solution u'_n and the exact solution u' in Example 4.2 ($n = 32$)

In Fig. 4-1 and Fig. 4-2 , the red dotted line is the numerical solution and the black line is the exact solution, it indicates that our presented method is very stable and effective. It is worth explaining that the method proposed In this chapter can not only be used to solve the nonlinear pulse problem, but also can be used to solve the linear problems.

Example 4.3 Consider the following impulsive differential equation with variable coefficients[36]

$$\beta u''(x) = 56x^6, x \in [0, 1] \setminus \{0.5\}, \quad \text{where} \quad \beta = \begin{cases} 1\ if\ \ 0 \le x \le 0.5 \\ 2\ if\ \ 0.5 < x \le 1. \end{cases}$$

subject to the boundary and interface conditions:

$$u(0) = 0, \quad u(1) = \frac{257}{512}, \quad \Delta u(0.5) = 0, \quad \Delta u'(0.5) = -0.5u'(0.5^-).$$

The exact solution

$$u(x) = \begin{cases} x^8 & x \in [0, 0.5] \\ \frac{1}{2}\left(x^8 + \frac{1}{256}\right) & x \in (0.5, 1] \end{cases}$$

Table 4-3: Comparison of absolute errors in Example 4.3

n	$\lvert u(x) - u_n(x)\rvert$ [36]	Presented method	
		max $\lvert u(x) - u_n(x)\rvert$	max $\lvert u'(x) - u'_n(x)\rvert$
20	1.975E-2	3.221E-3	2.849E-2
40	6.241E-3	7.207E-4	6.832E-3
80	1.743E-3	1.702E-4	1.669E-3
160	4.600E-4	4.132E-5	4.123E-4
320	1.181E-4	1.017E-5	1.024E-4
640	2.992E-5	2.524E-6	2.553E-5

4.5 Conclusion

In this chapter, combining the reproducing kernel method and the least square method to solve nonlinear impulsive differential equation, this method is proposed for the first time. A broken reproducing kernel space is cleverly built, the reproducing kernel space is reasonably simple because the author did not consider the complicated boundary conditions, and avoid the time consuming Schmidt orthogonalization process. The non-linear operator is transformed into a nonlinear optimization model, and the least square method is used to solve the problem. In fact, this technique can be extended to other class of impulsive boundary value problems. Although we just considered one pulse point in our presentation, by that analogy, the algorithm can also be applied to multiple pulse points. From the illustrative tables and figures, we obtain that the algorithm is remarkably accurate and effective as expected.

Chapter 5 The reproducing kernel method for Fredholm integro-differential equation

In this chapter, the simplified reproducing kernel method (SRKM for short) is presented to approximate the solution of second-order boundary value problems of Fredholm integro-differential equation. The convergence analysis of the method and the condition number of the matrix are also discussed. The proposed method is proved to be stable and have second order convergence. The algorithm is employed successfully in some numerical examples.

5.1 Fredholm integro-differential equation

In recent years, the integro-differential equations (IDEs for short) are an important branch of modern mathematics that arises naturally in different ares of applied biological phenomena[47], aeroelasticity phenomena[48], population dynamics[49], neural networks[50], and so on. The existence and uniqueness of the solutions for the higher-order IDEs have been investigated by Agarwal[51]; Unfortunately, it is difficult to obtain analytical solutions for the mentioned equations. So a numerical method is required. Iterative method[52] and normalized Bernstein polynomial method[53] are presented to solve boundary value problems of second-order IDEs. Wavelet method[54,55], walsh function method[56], Chebyshev finite difference method[57], differential transform method[58] and Legendre polynomial method[59] are also discussed for numerically solving IDEs. Multi-scale Galerkin method[60] and Finite Element method[61] are presented to approximate the solutions of second-order boundary value problems of Fredholm IDEs.

In this chapter, SRKM is developed to obtain stable numerical solutions of second-order boundary value problems of the Fredholm IDEs:

$$\begin{cases} u''(s) + p(s)u'(s) + q(s)u(s) + \lambda \displaystyle\int_0^1 k(s,t)u(t)dt = f(s), \quad s \in [0,1], \\ u(0) = \alpha, \quad u(1) = \beta. \end{cases} \tag{5-1}$$

where λ, α, β are real constants, $p(s), q(s)$ are two known functions, $f(s) \in L^2([0,1])$ and $k(s,t) \in C([0,1] \times [0,1])$ are given, and $u(s)$ is an unknown function to be determined.

As known to all, the reproducing kernel method is a powerful tool to solve the initial boundary value problems, and the SRKM is easy to get the approximate solution with higher precision. As to differential equations, the SRKM can reduce the condition number of the resulting discrete systems largely. In this chapter, we have established SRKM to solve the numerical solution of Eq. (5-1) in reproducing kernel space.

The rest of the paper is organized as follows. In section 2, we introduce the reproducing kernel space which will be used to discretize the IDEs. We present in section 3 the SRKM for solving second-order boundary value problems of Fredholm IDEs, and analyze the convergence and stability. In the section 4, the presented algorithms are applied to some numerical experiments. Then we end with some conclusions in section 5.

5.2 Preliminaries

Before the construction, it is necessary to present some notations, definitions, and preliminary facts upon the reproducing kernel theory that will be used further in the remainder of the paper. Throughout this chapter, $L^2[0,1] = \{u|\int_a^b u^2(x)dx < \infty\}$.

- Reproducing kernel space

Definition 5.1 The reproducing kernel space W_2^3 is defined as

$W_2^3[0,1] = \{u(x)|u''$ is an absolutely continuous real value function, $u''' \in L^2[0,1], u(0) = 0, u(1) = 0\}$. The inner product is given by

$$\langle u(x), v(x)\rangle = u(0)v(0) + u'(0)v'(0) + u''(0)v''(0) + \int_0^1 u'''v'''dx, \qquad u, v \in W_2^3[0,1].$$

Its reproducing kernel is $R_x(y)$, and the norm in W_2^3 is given by $\|u\| = \sqrt{\langle u, u\rangle_{W_2^3}}$.

Similarly, The reproducing kernel space W_2^1 is defined as

$W_2^1[0,1] = \{u(x)|u$ is an absolutely continuous real value function, $u' \in L^2[0,1]\}$. The inner product is given by

$$\langle u(x), v(x)\rangle = u(0)v(0) + \int_0^1 u'v'dx, \qquad u, v \in W_2^1[0,1].$$

Its reproducing kernel is $r_x(y)$, and the norm in W_2^1 is given by $\|u\| = \sqrt{\langle u, u\rangle_{W_2^1}}$.

- The initial conditions of homogeneous

Considering the characteristics of Eq. (5-1) with two initial conditions, for the convenience of describe our algorithm, put

$$v(s) = (1 - s)(u(s) - \alpha) + s(u(s) - \beta).$$

So, $v(0) = 0, v(1) = 0$, and $f(s)$ in Eq. (5-1) can also be reduced to a known function.

Therefore, Eq. (5-1) can be equivalently reduced to seeking out a function $u(t)$ satisfying the following equation:

$$\begin{cases} u''(s) + p(s)u'(s) + q(s)u(s) + \lambda \int_0^1 k(s,t)u(t)dt = f(s), \quad s \in [0,1], \\ u(0) = 0, \quad u(1) = 0. \end{cases} \tag{5-2}$$

On the other hand, the reproducing kernel space W_2^3 contains two conditions for $u(0) = 0, u(1) = 0$. So, we can solve Eq. (5-2) in W_2^3.

5.3 Description of the SRKM

In this section, we will develop the SRKM for solving Eq. (5-2) .

By Eq. (5-2) , we define a linear operator $\mathcal{L} : W_2^3[0,1] \to W_2^1[0,1]$:

$$\mathcal{L}u = u''(s) + p(s)u'(s) + q(s)u(s) + \lambda \int_0^1 k(s,t)u(t)dt, \quad u \in W_2^3[0,1]$$

It's easy to prove that $\mathcal{L}$ is bounded linear operator from W_2^3 into W_2^1.

So, Eq. (5-2) can be transformed into the following form:

$$\mathcal{L}u = f(s), \quad s \in [0,1]. \tag{5-3}$$

Suppose $\{x_i\}_{i=1}^{\infty}$ is dense on the interval $[0,1]$, put $\psi_i(x) = \mathcal{L}^* r_{x_i}(x)$, $i = 1, 2, \cdots$, where $\mathcal{L}^*$ is the adjoint operator of $\mathcal{L}$.

Theorem 5.1 $\psi_i(x) = \mathcal{L}R_x(x_i)$. $\quad i = 1, 2, \cdots$.

Proof: $\psi_i(x) = \langle \mathcal{L}^* r_{x_i}, R_x \rangle_{W_2^3} = \langle r_{x_i}, \mathcal{L}R_x \rangle_{W_2^1} = \mathcal{L}R_x(x_i)$. $i = 1, 2, \cdots$. □

From ref.[7], for each fixed n, it follows the function system $\{\psi_i(x)\}_{i=1}^n$ is linearly independent on W_2^3. Moreover, $\{\psi_i(x)\}_{i=1}^{\infty}$ is a complete system in the reproducing kernel space $W_2^3[0,1]$.

Let

$$S_n = span\{\{\psi_i(x)\}_{i=1}^n\}.$$

Then we can obtain that $S_n \subset W_2^3[0,1]$.

The orthogonal projection operator is denoted by $\mathcal{P}_n : W_2^3[0,1] \to S_n$, let $u_n \triangleq \mathcal{P}_n u$.

Theorem 5.2 If $u \in W_2^3[0,1]$ is the solution of Eq. (5-3) , then u_n satisfies the following:

$$\langle v, \psi_i \rangle = f(x_i) \quad i = 1, 2, \cdots, n. \tag{5-4}$$

Proof: Assume u is the solution of Eq. (5-3) , then

$$\langle u_n, \psi_i \rangle_{W_2^3} = \langle \mathcal{P}_n u, \psi_i \rangle_{W_2^3} = \langle u, \mathcal{P}_n \psi_i \rangle_{W_2^3} = \langle u, \psi_i \rangle_{W_2^3}$$

$$= \langle u, \mathcal{L}^* r_{x_i} \rangle_{W_2^3} = \langle \mathcal{L}u, r_{x_i} \rangle_{W_2^1} = \mathcal{L}u(x_i) = f(x_i). \qquad \square$$

In fact, x_n converges uniformly to x in W_2^3.

Theorem 5.3 If $u \in W_2^3[0,1]$ is the solution of Eq. (5-3) , then u_n uniformly converges to u.

Proof: $$|u_n(x) - u(x)| = |\langle u_n - u, R_t \rangle| \le \|R_x\|_{W_2^3} \ \|u_n - u\|_{W_2^3}$$
$$\le M\|u_n - u\|_{W_2^2} \to 0. \qquad \square$$

Similarly, we have the following conclusions.

Theorem 5.4 If $u \in W_2^3[0,1]$ is the solution of Eq. (5-3) , then $u_n^{(i)}$ uniformly converges to $u^{(i)}, i = 1, 2$.

According to the above discussion, the solution u_n of Eq. (5-5) is the approximate solution of Eq. (5-2) .

As $u_n \in S_n$, so

$$u_n(x) = \sum_{i=1}^{n} \lambda_i \psi_i(x) \tag{5-5}$$

To obtain the approximate solution u_n, we only need to obtain the coefficients of each $\psi_i(x)(i = 1, 2, \cdots, n)$. Use $\psi_i(x)$ to do the inner products with both sides of Eq. (5-5) , we have

$$\sum_{j=1}^{n} \lambda_j \langle \psi_i, \psi_j \rangle = f(x_i), \quad i = 1, 2, \cdots, n \tag{5-6}$$

This is the system of linear equations of $\lambda_i, i = 1, 2, \cdots, n$.

Let

$$G_n = \begin{bmatrix} \langle \psi_1, \psi_1 \rangle & \cdots & \langle \psi_1, \psi_n \rangle \\ \cdots & \cdots & \cdots \\ \langle \psi_n, \psi_1 \rangle & \cdots & \langle \psi_n, \psi_n \rangle \end{bmatrix}_{n \times n}$$

$$F = (f(x_1), f(x_2), \cdots, f(x_n))^T.$$

Consider that $\{\psi_i(t)\}_{i=1}^n$ is linearly independent in $W_2^3[0,1]$, so, G^{-1} is exist. Then, we have

$$(\lambda_1, \lambda_2, \cdots, \lambda_n)^T = G_n^{-1} \cdot F. \tag{5-7}$$

Theorem 5.5 The approximate solution u_n of Eq. (5-2) converges to its exact solution u with second order convergence.

Proof: It is known from the previous analysis that the solution of Eq. (5-2) is also the solution of $\mathcal{L}u = f$ in $W_2^3[0,1]$. Note that $W_2^3[0,1]$ is a reproducing kernel space, take advantage of theorem 4 in ref.[15], we have u_n converges to u with second order convergence. Therefore,

$$|u_n(x) - u(x)| \leq M\|u_n(x) - u(x)\|_{W_2^3} \leq M(M_1h^2) = M_2h^2.$$

where h is step-size on the interval $[0,1]$, M, M_1, M_2 are constants. □

Furthermore, the following rate of convergence formulas can be obtained:

$$\text{C.R}=log_2\frac{|u_n(x) - u(x)|}{|u_{2n}(x) - u(x)|}. \tag{5-8}$$

In addition to the convergence order analysis of the algorithm, the numerical stability of the algorithm is also considered. The following theorems show that the algorithm presented in this chapter is very stable and insensitive to errors in the calculation process.

Theorem 5.6 In Eq. (5-3) , if f has a disturbance variable δ, i.e. $\widetilde{f} = f + \delta$, it satisfies $\mathcal{L}\widetilde{u} = \widetilde{f}$, then $|u - \widetilde{u}| \leq M|\delta|$.

Proof: Considering the unique solution of the Eq. (5-3) , $\mathcal{L}$ is reversible, so

$$|u - \widetilde{u}| = |\mathcal{L}^{-1}u - \mathcal{L}^{-1}\widetilde{u}| = |\mathcal{L}^{-1}u - \mathcal{L}^{-1}(u+\delta)| = |\mathcal{L}^{-1}\delta| \leq \|\mathcal{L}^{-1}\|_{W_2^3}|\delta| \leq M|\delta|.$$ □

5.4 Numerical examples

In this section, the method proposed in this chapter is applied to some Fredholm IDEs to evaluate the approximate solution. In the example 5.1 and example 5.2 , the reproducing space is W_2^3. The present method is compared with the multiscale Galerkin method (MGM for short) and the finite element method (FEM for short). C.R stands for the computed convergence order of the approximate solutions, which is defined by Eq. (5-8) . Cond(A_n) denote the spectral condition numbers of the matrices A_n.

Example 5.1 Consider the following boundary value problem[60]

$$\begin{cases} u''(s) - \int_0^1 e^{st}u(t)dt = f(s), & s \in [0,1], \\ u(0) = u(1) = 0. \end{cases}$$

where $f(s) = 2 + \frac{(s-2)e^s + s + 2}{s^3}$. The exact solution of this problem is $u(s) = s(s-1)$.

The numerical results are given in table 5-1 . The results are compared with MGM. What is more, table 5-2 shows the result of adding the disturbance $\delta = 10^{-5}$ on the right-hand side f, it indicates that the disturbance is hardly affect the results of our method, this is consistent with the conclusion of theorem 5.6 . In Fig. 5-1 , the numerical solution (the red dots) is solved by the proposed method for $n = 15$, and the black line is the exact solution. All tables and figures show that the proposed approach is very stable and effective.

Table 5-1: The errors between numerical solutions and exact solution of Example 5.1

n	MGM [60]		Present method		
	$\|u_n - u\|$	C.R	$\mathrm{Max}\|u_n - u\|$	C.R	$\mathrm{Max}\|u'_n - u'\|$
15	3.6085E-2	–	9.1142E-5	–	1.1089E-4
31	1.8042E-2	1.0000	1.1144E-5	3.0318	1.5318E-5
63	9.0211E-3	1.0000	1.5890E-6	2.8101	2.6036E-6
127	4.5106E-3	1.0000	2.7249E-7	2.5438	8.9845E-7
255	2.2553E-3	1.0000	5.4165E-8	2.3308	2.5499E-7
511	1.1276E-3	1.0001	1.1904E-8	2.1859	6.7512E-8

Table 5-2: The absolute errors of added the disturbance 10^{-5} in Example 5.1

n	$\mathrm{Max}\|\widetilde{u_n} - u\|$	$\widetilde{\mathrm{C.R}}$	$\mathrm{Max}\|\widetilde{u'_n} - u'\|$
10	2.8676E-4	–	3.4648E-4
20	3.9560E-5	2.8577	4.9385E-5
40	5.1195E-6	2.9499	7.3729E-6
80	6.1431E-7	3.0589	3.2187E-6

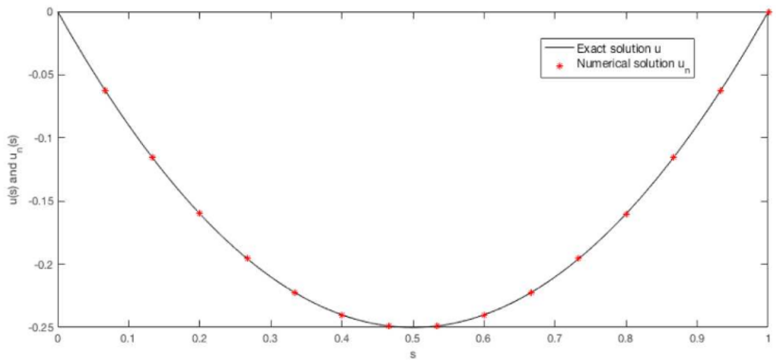

Fig. 5-1: The solutions in Example 5.1 ($n = 15$)

Example 5.2 Consider the following boundary value problem[60, 61]

$$\begin{cases} u''(s) + su'(s) + \pi^2 u(s) - \displaystyle\int_0^1 (s+t)u(t)dt = s\pi\cos(\pi s) - \dfrac{2s+1}{\pi}, \quad s \in [0,1], \\ u(0) = u(1) = 0. \end{cases}$$

The exact solution of Example 5.1 is $u(s) = \sin(\pi s)$.

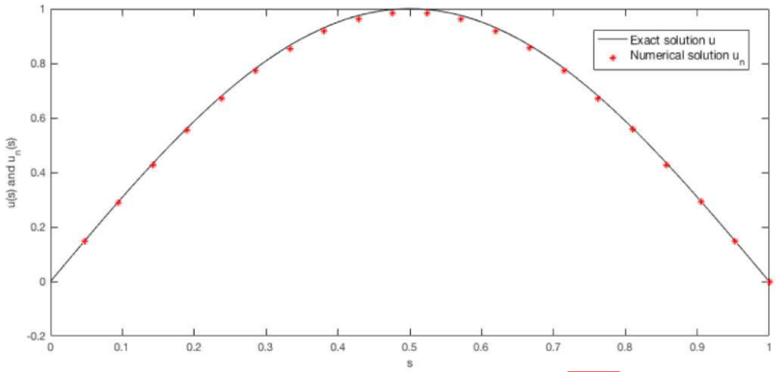

Fig. 5-2: The solutions in Example 5.2 ($n = 63$)

In table 5-3 , the results are compared with MGM and FEM. And it is found that these results are better than those obtained by MGM and FEM in higher resolution level. In Fig. 5-2 , the numerical solution (the red dots) is solved by the proposed method for $n = 63$, and the black line is the exact solution. Fig. 5-3 shows that the condition numbers of coefficient matrices are much smaller than these obtained by FEM. All the illustrative tables and figures show that the increase in the number of node results in a reduction in the absolute errors and correspondingly an improvement in the accuracy of the obtained solutions. This goes in agreement with the known fact, the error is decreasing, where more accurate solutions are achieved using an increase in the number of nodes.

Table 5-3: The errors between numerical solutions and exact solution in example 5.2

n	MGM [60]		FEM [61]	Present method			
	$\|u_n - u\|$	C.R	Cond(A_n)	Max$\|u_n - u\|$	C.R	Cond(A_n)	Max$\|u_n' - u'\|$
15	1.3311E-1	–	7.2411E+2	1.6421E-1	–	4.8047E+2	5.3499E-1
31	6.3906E-2	1.0586	2.9580E+3	4.9117E-2	1.7413	1.8042E+3	1.6259E-1
63	3.1599E-2	1.0161	1.1895E+4	1.2949E-2	1.9234	7.0018E+3	4.2968E-2
127	1.5754E-2	1.0041	4.7642E+4	3.2849E-3	1.9789	2.7597E+4	1.0903E-2
255	7.8715E-3	1.0010	1.9063E+5	8.2467E-4	1.9940	1.0959E+5	2.7372E-3
511	3.9350E-3	1.0003	7.6258E+5	2.0643E-4	1.9981	4.3679E+5	6.8519E-4
1023	1.9674E-3	1.0001	3.0504E+6	5.1632E-5	1.9994	1.7440E+6	1.7137E-4

5.5 Conclusion

In this chapter, a new algorithm for second-order Fredholm IDEs has been discussed. The SRKM is reasonably simple because the author used a uniform homogeneous reproducing kernel, and avoid the complex Schmidt orthogonalization process. The approxi-

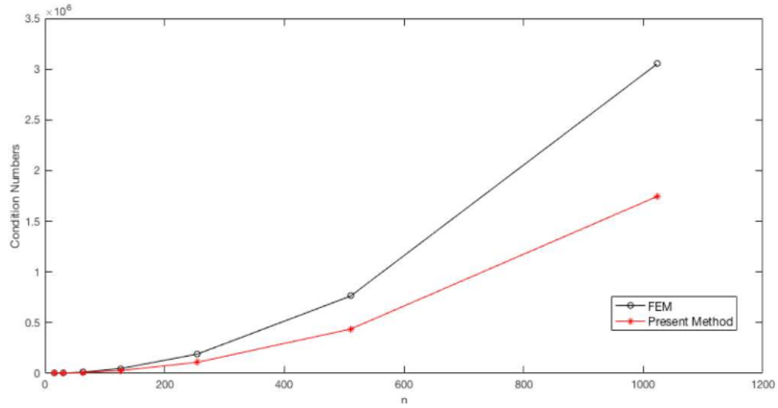

Fig. 5-3: A comparison of condition numbers of the coefficient matrices in Example 5.2

mate solutions obtained by this method and their derivative are both uniformly convergent with second order. The analysis of condition numbers and disturbance show that our algorithm is stable. Both theoretical analysis and numerical examples show that the proposed method is stable and can obtain optimal order of convergence.

Chapter 6 The reproducing kernel method for linear Volterra integral equations with variable coefficients

This chapter proposes a simplified reproducing kernel method to solve the linear Volterra integral equations with variable coefficients. The main idea of the method is to establish a reproducing kernel direct space that can be used in Volterra integral equations. And this chapter analyzes the convergence order and stability of the approximate solution. Then the uniform convergence of the numerical solution is proved, and the time consuming Schmidt orthogonalization process is avoided. The proposed method is proved to be stable and is not less than the second order convergence. The algorithm is proved to be feasible and stable through some numerical examples.

6.1 Linear Volterra integral equations

In this chapter, by simplified reproducing kernel method, we get an approximate solution for linear Volterra integral equations with variable coefficients as follows

$$\begin{cases} a_{11}(x)f_1(x) - b_{11}\int_0^x k_{11}(x,t)f_1(t)dt + a_{12}(x)f_2(x) - b_{12}\int_0^x k_{12}(x,t)f_2(t)dt = u_1(x) \\ a_{21}(x)f_1(x) - b_{21}\int_0^x k_{21}(x,t)f_1(t)dt + a_{22}(x)f_2(x) - b_{22}\int_0^x k_{22}(x,t)f_2(t)dt = u_2(x) \end{cases} \tag{6-1}$$

where $a_{ij}(x), i, j = 1, 2$ are arbitrary smooth functions defined on the interval [0,1], $b_{ij}, i, j = 1, 2$ are given constants.

The integral equations is an important branch of modern mathematics, many mathematical and physical problems need to be solved by integral equations or differential equations. The type of integral equations depending on the structure of integrals, for example, Fredholm integral equations, Volterra integral equations and Fredholm-Volterra integral equations. The model of Fredholm and Volterra integro-differential equations extends to every field of application, such as wind ripple in the desert, nano-hydrodynamics and drop wise condensation[62-64]. However, it is usually difficult to get an analytic solution of the integral and integro-differential equations, therefore, many researchers have extensively studied the numerical methods of Volterra integral equations in recent years[65–69]. F. Mirzaee[70] used the rationalized Haar functions to solve the system of linear Volterra integral equations. L.H. Yang[71, 72] provide a reproducing kernel method for solving the system of the Volterra integral equations. F. Mirzaee[73] solved the systems of linear Volterra integral equations based on the Euler matrix method. An expansion method is used for treatment of second kind Volterra integral equations system[74]. E. Hesameddini[75] solved the Volterra-Fredholm integral equations based on Bernstein polynomials and hybrid Bernstein Block-Pulse functions. F. Mirzaee[76] contributes an efficient numerical approach to solve the systems of high-order linear Volterra integro-differential equations with variable coefficients under the mixed conditions.

As known to all, the application of reproducing kernel method for integral and differential equations has been developed by many researchers because this method is easy to obtain the exact solution with the series form and to get approximate solution with higher precision[7, 15, 41]. Moreover, more and more scholars use the reproducing kernel method to solve the problem of integral-differential equations[71, 72]. The traditional reproducing kernel method is very complicated because it contains Schmidt orthogonalization process. The simplified regenerative kernel method proposed in this chapter avoids Schmidt's orthogonalization process and eliminates need to calculate individual reproducing kernel functions, Which makes it more widely applicable.

The aim of this chapter is to derive the numerical solutions of Eqs. (6-1) in section 1. In section 2, we introduce the reproducing kernel direct space for solving problems. Some primary results are analyzed in section 3. The numerical algorithm of approximate solution is presented in section 4. Section 5 describes the convergence order and stability analysis of approximate solution. In the section 6, the presented algorithms are applied to some numerical experiments. Then we end with some conclusions in section 7.

6.2 Reproducing kernel direct space

In this section, the reproducing kernel space is given, and the reproducing kernel direct space is defined that we need. We assume that Eqs. (6-1) have the unique solution.

Reproducing kernel space $W_2[0,1]$ is defined as：

$$W_2[0,1] = \{u(x) | u' \text{ is an absolutely continuous real value funcion in}[0,1], u'' \in L^2[0,1]\}$$

The inner product and norm are given by ref.[7].

Reproducing kernel space $W_1[0,1]$ is defined as:

$$W_1[0,1] = \{u(x)|u \text{ is an absolutely continuous real value funcion in}[0,1], u' \in L^2[0,1]\}$$

The inner product and norm are given by ref.[7].

The reproducing kernel spaces are W_2 and W_1 with reproducing kernel $R_t(x)$ and $r_t(x)$, respectively.

In this chapter, consider that the exact solution of Eqs. (6-1) is a function vector, so, we structure a reproducing kernel direct space, introduce product and norm.

Definition 6.1 The linear space $W_{(2,2)}$ is defined as

$$W_{(2,2)}[0,1] = W_2[0,1] \oplus W_2[0,1] = \{\mathbf{F}(x) = (f_1(x), f_2(x))^T | f_1(x), f_2(x) \in W_2[0,1]\}$$

The inner product and norm are defined by

$$\langle \mathbf{F}(x), \mathbf{G}(x)\rangle_{W_{(2,2)}} = \langle f_1(x), g_1(x)\rangle_{W_2} + \langle f_2(x), g_2(x)\rangle_{W_2}$$

$$||\mathbf{F}(x)||^2_{W_{(2,2)}} = ||f_1(x)||^2_{W_2} + ||f_2(x)||^2_{W_2}$$

Theorem 6.1 The space $W_{(2,2)}[0,1]$ is a Hilbert space.

Proof: Suppose that $\{\mathbf{F}_n(x)\}_{n=1}^{\infty}$ is a Cauchy sequence in $W_{(2,2)}[0,1]$, however,

$$\mathbf{F}_n(x) = (f_{1,n}(x), f_{2,n}(x))^T, n = 1, 2, \cdots.$$

so, $\{f_{1,n}(x)\}_{n=1}^{\infty}$ and $\{f_{2,n}(x)\}_{n=1}^{\infty}$ are Cauchy sequences in W_2, respectively.

Notice that W_2 is a reproducing kernel space, so, there are two functions $g_1(x), g_2(x) \in W_2$, make

$$||f_{1,n}(x) - g_1(x)||^2_{W_2} \to 0, \quad ||f_{2,n}(x) - g_2(x)||^2_{W_2} \to 0.$$

Let

$$\mathbf{G}(x) = (g_1(x), g_2(x))^T.$$

By the definition 6.1 , $\mathbf{G}(x) \in W_{(2,2)}[0,1]$, and

$$||\mathbf{F}_n(x) - \mathbf{G}(x)||^2_{W_{(2,2)}} = ||f_{1,n}(x) - g_1(x)||^2_{W_2} + ||f_{2,n}(x) - g_2(x)||^2_{W_2} \to 0.$$

So, the space $W_{(2,2)}[0,1]$ is a Hilbert space, we call it the reproducing kernel direct space. □

Similarly, the reproducing kernel direct space $W_{(1,1)}[0,1]$ is defined as

$$W_{(1,1)}[0,1] = W_1[0,1] \oplus W_1[0,1]$$
$$= \{\mathbf{U}(x) = (u_1(x), u_2(x))^T | u_1(x), u_2(x) \in W_1[0,1]\}$$

In order to solve Eqs. (6-1) , we introduce a linear operator $\mathcal{L} : W_{(2,2)}[0,1] \rightarrow W_{(1,1)}[0,1]$,

$$\mathcal{L}\mathbf{F} = \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \quad \text{others } \mathbf{F} = (f_1, f_2)^T$$

and

$$\begin{cases} \mathcal{L}_{11} f_1 = a_{11} f_1(x) - b_{11} \int_0^x k_{11}(x,t) f_1(t) dt \\ \mathcal{L}_{12} f_2 = a_{12} f_2(x) - b_{12} \int_0^x k_{12}(x,t) f_2(t) dt \\ \mathcal{L}_{21} f_1 = a_{21} f_1(x) - b_{21} \int_0^x k_{21}(x,t) f_1(t) dt \\ \mathcal{L}_{22} f_2 = a_{22} f_2(x) - b_{22} \int_0^x k_{22}(x,t) f_2(t) dt \end{cases}$$

By Ref.[7], it's easy to prove that $\mathcal{L}$ is a bounded operator.

Then, Eqs. (6-1) is equivalent to the operator equation in $W_{(2,2)}[0,1]$:

$$\mathcal{L}\mathbf{F} = \mathbf{U} \tag{6-2}$$

6.3 Basic properties

In this section, the approximate solution of Eqs. (6-2) is presented.

Take $\{x_i\}_{i=1}^{\infty}$ is dense on the interval $[0,1]$, put

$$\phi_{ijk}(x) = \mathcal{L}_{ij}^* r_{x_k}(x). \quad i,j = 1,2, \quad k = 1,2,\cdots$$

where $\mathcal{L}_{ij}^*$ is the adjoint operator of $\mathcal{L}_{ij}$.

Theorem 6.2 $\phi_{ijk}(x) = \mathcal{L}_{ij} R_x(x_k). \quad i,j = 1,2, \quad k = 1,2,\cdots$

Proof: From the properties of the reproducing kernel function, it can be inferred that

$$\begin{aligned} \phi_{ijk}(x) &= \langle \mathcal{L}_{ij}^* r_{x_k}, R_x \rangle_{W_2} \\ &= \langle r_{x_k}, \mathcal{L}_{ij} R_x \rangle_{W_1} \\ &= \mathcal{L}_{ij} R_x(x_k) \end{aligned}$$

□

Put

$$\psi_{i1}(x) = (\phi_{11i}(x), \phi_{12i}(x))^T, \quad \psi_{i2}(x) = (\phi_{21i}(x), \phi_{22i}(x))^T. \quad i = 1,2,\cdots$$

Theorem 6.3 For each fixed n, $\{\psi_{ij}\}_{(1,1)}^{(n,2)}$ is linearly independent in $W_{(2,2)}[0,1]$.

Proof: Let $0=\sum_{i=1}^{n}\sum_{j=1}^{2}C_{ij}\psi_{ij}(t)$.

Consider $u_{k,1}(t)\in W_1[0,1]$, make

$$u_{k,1}(t)\begin{cases}=0, & t=x_1,x_2,\cdots,x_{k-1},x_{k+1},\cdots,x_n\\ \neq 0, & \text{others}\end{cases}$$

and $W_1[0,1]\ni u_{k,2}(t)=0$. Put $\mathbf{U}_k(t)=(u_{k,1}(t),u_{k,2}(t))^T$, obviously, $\mathbf{U}_k(t)\in W_{(1,1)}[0,1]$.

Take $\mathbf{F}_k\in W_{(2,2)}[0,1]$, make $\mathcal{L}\mathbf{F}_k=\mathbf{U}_k$. So,

$$\begin{aligned}
0&=\langle \mathbf{F}_k,\sum_{i=1}^{n}\sum_{j=1}^{2}C_{ij}\psi_{ij}\rangle\\
&=\sum_{i=1}^{n}(C_{i1}\langle \mathbf{F}_k,\psi_{i1}\rangle+C_{i2}\langle \mathbf{F}_k,\psi_{i2}\rangle)\\
&=\sum_{i=1}^{n}(C_{i1}(\langle f_{k,1},\phi_{11i}\rangle+\langle f_{k,2},\phi_{12i}\rangle)+C_{i2}(\langle f_{k,1},\phi_{21i}\rangle+\langle f_{k,2},\phi_{22i}\rangle))\\
&=\sum_{i=1}^{n}(C_{i1}(\langle f_{k,1},\mathcal{L}_{11}^{*}r_{x_i}\rangle+\langle f_{k,2},\mathcal{L}_{12}^{*}r_{x_i}\rangle)+C_{i2}(\langle f_{k,1},\mathcal{L}_{21}^{*}r_{x_i}\rangle+\langle f_{k,2},\mathcal{L}_{22}^{*}r_{x_i}\rangle))\\
&=\sum_{i=1}^{n}(C_{i1}(\langle \mathcal{L}_{11}f_{k,1},r_{x_i}\rangle+\langle \mathcal{L}_{12}f_{k,2},r_{x_i}\rangle)+C_{i2}(\langle \mathcal{L}_{21}f_{k,1},r_{x_i}\rangle+\langle \mathcal{L}_{22}f_{k,2},r_{x_i}\rangle))\\
&=\sum_{i=1}^{n}(C_{i1}(\mathcal{L}_{11}f_{k,1}(x_i)+\mathcal{L}_{12}f_{k,2}(x_i))+C_{i2}(\mathcal{L}_{21}f_{k,1}(x_i)+\mathcal{L}_{22}f_{k,2}(x_i)))\\
&=\sum_{i=1}^{n}(C_{i1}u_{k,1}(x_i)+C_{i2}u_{k,2}(x_i))\\
&=C_{k1}u_{k,1}(x_k)
\end{aligned}$$

So,

$$C_{k,1}=0,\quad k=1,2,\cdots,n$$

Similarly, we have $C_{k,2}=0,\quad k=1,2,\cdots,n$. □

Theorem 6.4 $\{\psi_{ij}\}_{(1,1)}^{(\infty,2)}$ is complete in space $W_{(2,2)}[0,1]$.

Proof: For each $\mathbf{F}(x)=(f_1(x),f_2(x))^T\in W_{(2,2)}[0,1]$, it follows that $\langle \mathbf{F}(x),\psi_{ij}(x)\rangle=0$ for every $i=1,2,\cdots,j=1,2$. So,

$$\begin{aligned}
0&=\langle \mathbf{F}(x),\psi_{i1}(x)\rangle_{W_{(2,2)}}\\
&=\langle f_1(x),\mathcal{L}_{11}^{*}r_{x_i}(x)\rangle_{W_2}+\langle f_2(x),\mathcal{L}_{12}^{*}r_{x_i}(x)\rangle_{W_2}\\
&=\langle \mathcal{L}_{11}f_1(x),r_{x_i}(x)\rangle_{W_1}+\langle \mathcal{L}_{12}f_2(x),r_{x_i}(x)\rangle_{W_1}\\
&=\mathcal{L}_{11}f_1(x_i)+\mathcal{L}_{12}f_2(x_i)
\end{aligned}$$

Similarly, we have $\mathcal{L}_{21}f_1(x_i) + \mathcal{L}_{22}f_2(x_i) = 0$. This is equivalent to $\mathcal{L}\mathbf{F} = 0$, Since Eqs. (6-2) has a unique solution. It follows that

$$\mathbf{F} = 0$$

□

Let $S_n = span\{\psi_{ij}\}_{(1,1)}^{(n,2)}$. By the Theorem 6.2 and Theorem 6.3 , then one can obtain that $S_n \subset W_{(2,2)}[0, 1]$.

The orthogonal projection operator is denoted by $\mathcal{P}_n : W_{(2,2)}[0, 1] \to S_n$. Then we obtain a theorem as follows, which is of great significance to us.

Theorem 6.5 If $\mathbf{F} \in W_{(2,2)}[0, 1]$ is the solution of Eqs. (6.1) , then $\mathbf{F}_n = \mathcal{P}_n\mathbf{F}$ satisfies the following

$$\langle \mathbf{V}, \psi_{ij}\rangle_{W_{(2,2)}} = u_j(x_i), \quad i = 1, 2, \cdots, n, \quad j = 1, 2 \tag{6-3}$$

Proof: It can be proven that substituting $\mathbf{F}_n$ into Eq. (6-3) holds true, in fact

$$\begin{aligned}
\langle \mathcal{P}_n\mathbf{F}, \psi_{i1}\rangle_{W_{(2,2)}} = \langle \mathbf{F}, \mathcal{P}_n\psi_{i1}\rangle_{W_{(2,2)}} &= \langle \mathbf{F}, \psi_{i1}\rangle_{W_{(2,2)}} \\
&= \langle f_1, \mathcal{L}_{11}^* r_{x_i}\rangle_{W_2} + \langle f_2, \mathcal{L}_{12}^* r_{x_i}\rangle_{W_2} \\
&= \mathcal{L}_{11}f_1(x_i) + \mathcal{L}_{12}f_2(x_i) \\
&= u_1(x_i)
\end{aligned}$$

Similarly, we have

$$\langle \mathcal{P}_n\mathbf{F}, \psi_{i2}\rangle_{W_{(2,2)}} = u_2(x_i)$$

□

In fact, $\mathbf{F}_n(x)$ is an approximate solution of Eqs. (6-2) .

Theorem 6.6 If $\mathbf{F} \in W_{(2,2)}[0, 1]$ is the solution of Eqs. (6-2) , then $\mathbf{F}_n$ converges uniformly to $\mathbf{F}$.

Proof: From the properties of the reproducing kernel function, it can be inferred that

$$\begin{aligned}
|f_1(t) - f_{1,n}(t)| &= |\langle f_1 - f_{1,n}, R_t\rangle| \\
&\leq \|f_1 - f_{1,n}\|_{W_2} \ \|R_t\|_{W_2} \\
&\leq M \ \|f_1 - f_{1,n}\|_{W_2}
\end{aligned}$$

Similarly, we have

$$|f_2(t) - f_{2,n}(t)| \leq M \ \|f_2 - f_{2,n}\|_{W_2}$$

–

6.4 Numerical algorithm

In this section, the numerical algorithm for the approximate solution $\mathbf{F}_n$ is given. By the knowledge of previous section, the exact solution of Eqs. (6-2) can be expressed as

$$\mathbf{F}(x) = \sum_{i=1}^{\infty}\sum_{j=1}^{2} C_{ij}\psi_{ij}(x) \tag{6-4}$$

As $\mathbf{F}_n \subset S_n \subset W_{(2,2)}[0,1]$, so, we obtain the approximate solution of Eqs. (6-2)

$$\mathbf{F}_n(x) = \sum_{i=1}^{n}\sum_{j=1}^{2} C_{ij}\psi_{ij}(x) \tag{6-5}$$

To obtain the approximate solution $\mathbf{F}_n$, we only need to obtain the coefficients of each $\psi_{ij}(x)$. Use $\psi_{ij}(x)$ to do the inner products with both sides of Eqs. (6-5) , by the Theorem 6.5 , we have

$$\begin{cases} \sum_{i=1}^{n} C_{i1}\langle\psi_{i1},\psi_{m1}\rangle + \sum_{j=1}^{n} C_{j2}\langle\psi_{j2},\psi_{m1}\rangle = u_1(x_m), \quad m = 1,2,\cdots,n \\ \sum_{i=1}^{n} C_{i1}\langle\psi_{i1},\psi_{m2}\rangle + \sum_{j=1}^{n} C_{j2}\langle\psi_{j2},\psi_{m2}\rangle = u_2(x_m), \quad m = 1,2,\cdots,n \end{cases} \tag{6-6}$$

Eqs. (6-6) is the system of linear equations of $C_{ij}, i = 1,2,\cdots,n, \quad j = 1,2.$

Let

$$G_{2n} = \begin{bmatrix} \langle\psi_{i1},\psi_{m1}\rangle & \cdots & \langle\psi_{j2},\psi_{m1}\rangle \\ \cdots & \cdots & \cdots \\ \langle\psi_{i1},\psi_{m2}\rangle & \cdots & \langle\psi_{j2},\psi_{m2}\rangle \end{bmatrix}_{i,j,k=1,2,\cdots,n}$$

$$B = (u_1(x_1),\cdots,u_1(x_n),u_2(x_1),\cdots,u_2(x_n))^T$$

Consider that $\{\psi_{ij}(x)\}_{(1,1)}^{(n,2)}$ is linearly independent in $W_{(2,2)}[0,1]$, so, G^{-1} is exist. Then, we have

$$(C_{11},C_{12},\cdots,C_{1n},C_{21},C_{22},\cdots,C_{2n})^T = G_{2n}^{-1}\cdot B$$

6.5 Convergence order and stability analysis

In this section, the convergence order of the approximate solution is proved, and we discussed the stability of approximate solution.

Theorem 6.7 The approximate solution $\mathbf{F}_n$ of Eqs. (6-2) converges to its exact solution $\mathbf{F}$ with not less than the second order convergence.

Proof: The subtraction of the two equations in Eqs. (6-1) can obtain the following form

$$a(x)f_1(x) - \int_0^x k(x,t)f_1(t)dt = u(x) \tag{6-7}$$

For each identified $f_2(x)$, $u(x), a(x), k(x,t)$ in Eqs. (6-7) are known functions. Note that Eqs. (6-7) is a linear operator equation $\mathcal{L}f_1 = u$ in reproducing kernel space $W_2[0,1]$, take advantage of Theorem 4 in [15], we have $f_{1,n}$ converges to f_1 not less than the second order convergence. Therefore

$$|f_1(x) - f_{1,n}(x)| \leq M \ \|f_1 - f_{1,n}\|_{W_2} \leq M_1 h^2.$$

In this chapter, we describe the approximation of $\mathbf{F}$ and $\mathbf{F}_n$ by Euclidean distance as the following form:

$$d(\mathbf{F},\mathbf{F}_n) = \sqrt{\max_x |f_1(x) - f_{1,n}(x)|^2 + \max_x |f_2(x) - f_{2,n}(x)|^2}$$

So,

$$d(\mathbf{F},\mathbf{F}_n) \leq \sqrt{(M_1h^2)^2 + (M_2h^2)^2} = M_3h^2$$

where h is step-size on the interval $[0,1]$, M, M_1, M_2, M_3 are constants. □

Furthermore, the following rate of convergence formulas can be obtained:

$$\text{C.R}_1 = \log_2 \frac{|f_1(x) - f_{1,n}(x)|}{|f_1(x) - f_{1,2n}(x)|}$$

$$\text{C.R}_2 = \log_2 \frac{|f_2(x) - f_{2,n}(x)|}{|f_2(x) - f_{2,2n}(x)|}$$

The convergence order based on Euclidean distance is defined as follows:

$$\text{C.R}_E = \log_2 \frac{d(\mathbf{F},\mathbf{F}_n)}{d(\mathbf{F},\mathbf{F}_{2,n})}$$

In addition to the convergence order analysis of the algorithm, the numerical stability of the algorithm is also considered. The following theorems show that the algorithm presented in this chapter is very stable and insensitive to errors in the calculation process.

Theorem 6.8 In Eqs. (6-2) , if $\mathbf{U}$ has a disturbance variable δ, i.e. $\widetilde{\mathbf{U}} = \mathbf{U} + \delta$, it satisfies $\mathcal{L}\widetilde{\mathbf{F}} = \widetilde{\mathbf{U}}$, then

$$\|\mathbf{F} - \widetilde{\mathbf{F}}\|_{W(2,2)} \leq M\|\delta\|_{W(2,2)}$$

Proof: Considering the unique solution of the Eqs. (6-2) , $\mathcal{L}$ is reversible, so

$$\begin{aligned}||\mathbf{F}-\widetilde{\mathbf{F}}||_{W_{(2,2)}} &= ||\mathcal{L}^{-1}\mathbf{U}-\mathcal{L}^{-1}\widetilde{\mathbf{U}}||_{W_{(2,2)}} \\ &= ||\mathcal{L}^{-1}\mathbf{U}-\mathcal{L}^{-1}(\mathbf{U}+\delta)||_{W_{(2,2)}} \\ &= ||\mathcal{L}^{-1}\delta||_{W_{(2,2)}} \\ &\le ||\mathcal{L}^{-1}||_{W_{(2,2)}}\ \ ||\delta||_{W_{(2,2)}} \\ &\le M||\delta||_{W_{(2,2)}}\end{aligned}$$

□

6.6 Numerical examples

In this section, the method proposed in this chapter is applied to some linear Volterra equations to evaluate the approximate solution. We compare the numerical results with the other methods discussed in ref.[72, 73, 77]. Finally, the results show that our algorithm is practical and remarkably effective.

Example 6.1 Consider the linear Volterra equations with variable coefficients[72]

$$\begin{cases} 2xf_1(x)-\int_0^x 3tf_1(t)dt+xf_2(x)-\int_0^x(2x+1)f_2(t)dt=u_1(x) \\ xf_1(x)-\int_0^x 2(x+t)f_1(t)dt-2xf_2(x)-\int_0^x 2(x+t)tf_2(t)dt=u_2(x)\end{cases}$$

with $u_1(x)=2x$, $u_2(x)=x-\frac{5x^3}{3}+\frac{7x^4}{6}$. The exact solution is $\mathbf{F}(x)=(x+1,-x)^T$.

Table 6-1: The absolute errors of $f_1(x)$ in Example 6.1

x	Absolute error [72]		Present method	
	$\lvert f_2(x)-f_{2,10}(x)\rvert$	$\lvert f_2(x)-f_{2,50}(x)\rvert$	$\lvert f_2(x)-f_{2,10}(x)\rvert$	$\lvert f_2(x)-f_{2,50}(x)\rvert$
0.1	1.6526E-2	8.0353E-4	9.8627E-4	7.9088E-6
0.2	3.7037E-3	1.6556E-5	4.3885E-4	2.5633E-6
0.3	5.5313E-4	8.4614E-6	1.8281E-4	4.6623E-7
0.4	6.1346E-4	6.9591E-5	3.8616E-5	2.8473E-6
0.5	1.5801E-3	8.9074E-5	2.3524E-4	4.9175E-6
0.6	3.2951E-3	5.9273E-5	4.1948E-4	6.7655E-6
0.7	4.7018E-3	7.2176E-5	5.9114E-4	8.4060E-6
0.8	5.7254E-3	8.3177E-5	7.4888E-4	9.8286E-6
0.9	6.6785E-3	9.2161E-5	8.9028E-4	1.1016E-5
1.0	7.6319E-3	9.9042E-5	1.0131E-3	1.1957E-5

Table 6-2: The absolute errors of $f_2(x)$ in Example 6.1

x	Absolute error [72]		Present method	
	$\lvert f_2(x)-f_{2,10}(x)\rvert$	$\lvert f_2(x)-f_{2,50}(x)\rvert$	$\lvert f_2(x)-f_{2,10}(x)\rvert$	$\lvert f_2(x)-f_{2,50}(x)\rvert$
0.1	1.6526E-2	8.0353E-4	9.8627E-4	7.9088E-6
0.2	3.7037E-3	1.6556E-5	4.3885E-4	2.5633E-6
0.3	5.5313E-4	8.4614E-6	1.8281E-4	4.6623E-7
0.4	6.1346E-4	6.9591E-5	3.8616E-5	2.8473E-6
0.5	1.5801E-3	8.9074E-5	2.3524E-4	4.9175E-6
0.6	3.2951E-3	5.9273E-5	4.1948E-4	6.7655E-6
0.7	4.7018E-3	7.2176E-5	5.9114E-4	8.4060E-6
0.8	5.7254E-3	8.3177E-5	7.4888E-4	9.8286E-6
0.9	6.6785E-3	9.2161E-5	8.9028E-4	1.1016E-5
1.0	7.6319E-3	9.9042E-5	1.0131E-3	1.1957E-5

Table 6-3: Comparison of absolute errors and convergence in Example 6.1

n	$\max\lvert f_1-f_{1,n}\rvert$	$C.R_1$	$\max\lvert f_2-f_{2,n}\rvert$	$C.R_2$	$C.R_E$
10	2.1255E-3	-	1.0131E-3	-	-
20	4.3934E-4	2.2	2.2560E-4	2.1	2.2
40	1.0316E-4	2.1	5.2831E-5	2.1	2.1
80	2.5038E-5	2.0	1.2697E-5	2.1	2.0

Employing the simplified reproducing kernel method, we obtain the numerical results are given in Table 6-1 and Table 6-2 taking n=10,50. Lihong Yang et al[72] use the Schmidt orthogonalization method to solve the linear Volterra equations in the reproducing kernel space, this method is complex in computation, and the convergence order and stability are not as good as the present method. In Table 6-3 , we compared the absolute errors and the order of convergence, it indicates that our method is more accurate than the traditional reproducing kernel method, and the approximate solution has not less than the second order convergence.

Example 6.2 Consider the following linear Volterra equations[72]

$$\begin{cases} f_1(x)-\int_0^x (t^2-x)f_1(t)dt-\int_0^x (t^2-x)f_2(t)dt = x+\frac{1}{2}x^3+\frac{1}{12}x^4-\frac{1}{5}x^5 \\ -\int_0^x tf_1(t)dt+f_2(x)-\int_0^x tf_2(t)dt = x^2-\frac{1}{3}x^3-\frac{1}{4}x^4 \end{cases}$$

and exact solution $\mathbf{F}(x) = (x, x^2)^T$.

Table 6-4: Absolute errors for Example 6.2

x	Absolute error[72]		Present method	
	$\lvert f_1(x) - f_{1,50}(x)\rvert$	$\lvert f_2(x) - f_{2,50}(x)\rvert$	$\lvert f_1(x) - f_{1,50}(x)\rvert$	$\lvert f_2(x) - f_{2,50}(x)\rvert$
0.1	2.2500E-7	4.0839E-5	2.2552E-7	8.3705E-9
0.2	1.3644E-6	6.7404E-6	4.4246E-7	3.1843E-9
0.3	2.4508E-6	1.0417E-5	6.4908E-7	1.0682E-8
0.4	4.6226E-6	1.4523E-5	8.4201E-7	3.7710E-8
0.5	7.3781E-6	1.9340E-5	1.0192E-6	8.2320E-8
0.6	1.1063E-5	2.5027E-5	1.1800E-6	1.4912E-7
0.7	1.5766E-5	3.2174E-5	1.3247E-6	2.4314E-7
0.8	2.2068E-5	4.1132E-5	1.4549E-6	3.7012E-7
0.9	3.0466E-5	4.2400E-5	1.5738E-6	5.3653E-7
1.0	4.4805E-5	6.5250E-5	1.6823E-6	1.1336E-6

Table 6-5: Comparison of absolute errors and convergence in Example 6.2

n	$\max\lvert f_1 - f_{1,n}\rvert$	$C.R_1$	$\max\lvert f_2 - f_{2,n}\rvert$	$C.R_2$	$C.R_E$
10	1.8831E-4	-	1.2393E-4	-	-
20	2.5224E-5	2.9	1.6849E-5	2.8	2.9
40	3.2595E-6	2.9	2.1951E-6	2.9	2.9
80	3.8726E-7	3.0	2.7438E-7	3.0	3.0

Table 6-6: The absolute errors of added the disturbance 10^{-5} in Example 6.2

n	$\max\lvert f_1 - \widetilde{f}_{1,n}\rvert$	$\widetilde{C.R}_1$	$\max\lvert f_2 - \widetilde{f}_{2,n}\rvert$	$\widetilde{C.R}_2$	$\widetilde{C.R}_E$
10	1.8577E-4	-	1.4270E-4	-	-
20	2.3000E-5	3.0	3.5622E-5	2.0	2.5

Table 6-4 shows the numerical results and comparison with the others methods(Ref.[72]). Table 6-5 shows the numerical results and the convergence order with different n. Table 6-6 shows the result of adding the disturbance 10^{-5} on the right-hand side $\mathbf{U}$, it indicates that the disturbance is hardly affect the results of our method. All tables show that the proposed approach is very stable and effective.

Example 6.3 For the last example, consider the following Volterra equations[73, 77]

$$\begin{cases} f_1(x) - \int_0^x (\sin(x-t) - 1) f_1(t)dt - \int_0^x (1 - t\cos x) f_2(t)dt = u_1(x) \\ -\int_0^x f_1(t)dt + f_2(x) - \int_0^x (x-t) f_2(t)dt = u_2(x) \end{cases}$$

$u_1(x)$ and $u_2(x)$ are chosen such that the exact solution is $\mathbf{F}(x) = (cosx, sinx)^T$.

Table 6-7: Comparison of the absolute errors in Example 6.3

x	$e(f_1)$[73]	$e(f_2)$[73]	$e(f_1)$[77]	$e(f_2)$[77]	Present method ($n = 80$)	
					$\lvert f_1 - f_{1,n}\rvert$	$\lvert f_2 - f_{2,n}\rvert$
0.2	2.0571E-7	7.2964E-7	2.0571E-7	7.2964E-7	2.9204E-8	6.5880E-7
0.4	2.3189E-6	6.5961E-7	2.3189E-6	6.5961E-7	3.7743E-8	7.3077E-7
0.6	3.8191E-5	3.7139E-6	3.8191E-5	3.7140E-6	4.6379E-8	7.7264E-7
0.8	2.8744E-4	2.9571E-5	2.8744E-4	2.9572E-5	3.9317E-8	7.8413E-7
1	1.3807E-3	1.7559E-4	1.3807E-3	1.7560E-4	2.0401E-9	7.4482E-7

Table 6-8: Comparison of absolute errors and convergence in Example 6.3

n	$\max \lvert f_1 - f_{1,n}\rvert$	C.R$_1$	$\max \lvert f_2 - f_{2,n}\rvert$	C.R$_2$	C.R$_E$
10	6.4180E-5	-	6.4179E-5	-	-
20	4.2627E-6	3.9	4.2627E-6	2.8	2.8
40	3.1270E-7	3.7	3.1270E-7	2.9	2.9
80	4.6944E-8	2.7	4.6944E-8	2.9	2.9

Numerical results in Example 6.3 are given in Tables 6-7 and 6-8 . Table 6-7 shows the comparison of the numerical results of the absolute error functions obtained by the present method, the Euler matrix method[73] and the collocation method [77] for n=80. Fig. 6-1 shows the errors $f_{1,n}(x) - f_1(x)$ in different situations, Fig. 6-2 shows the semi logarithmic coordinates of absolute errors $e_{2,n}(x) = |f_{2,n}(x) - f_2(x)|$ in four cases. All of tables and figures show that our method converges rapidly.

6.7 Summary

In this chapter, a simplified reproducing kernel method for linear Volterra integral equations, the convergence order and stability of the approximate solution are analyzed for

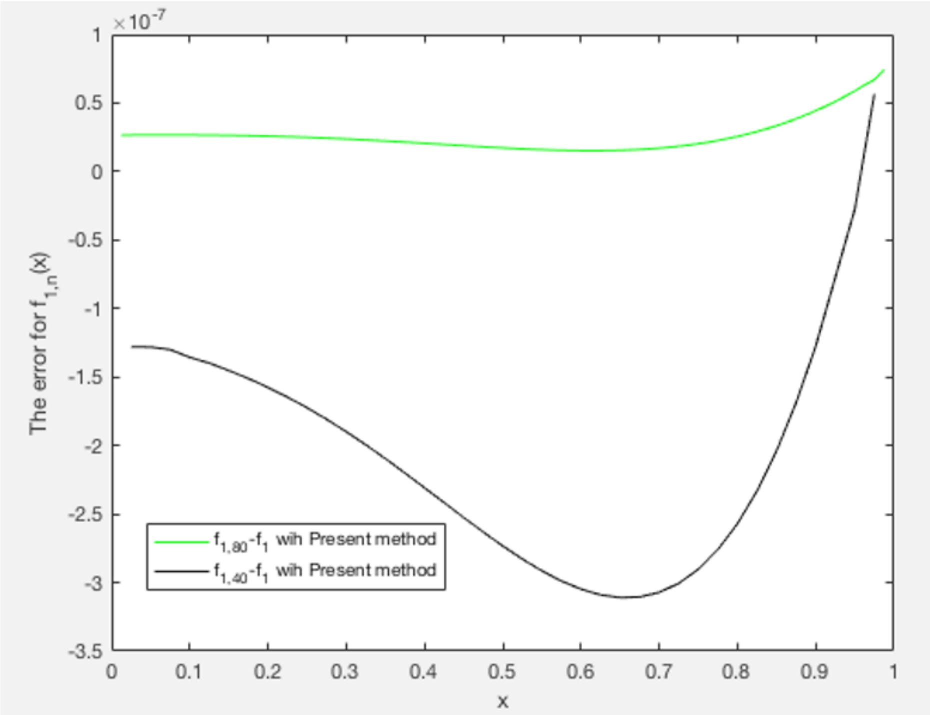

Fig. 6-1: Error comparison of function of $f_1(x)$ in Example 6.3

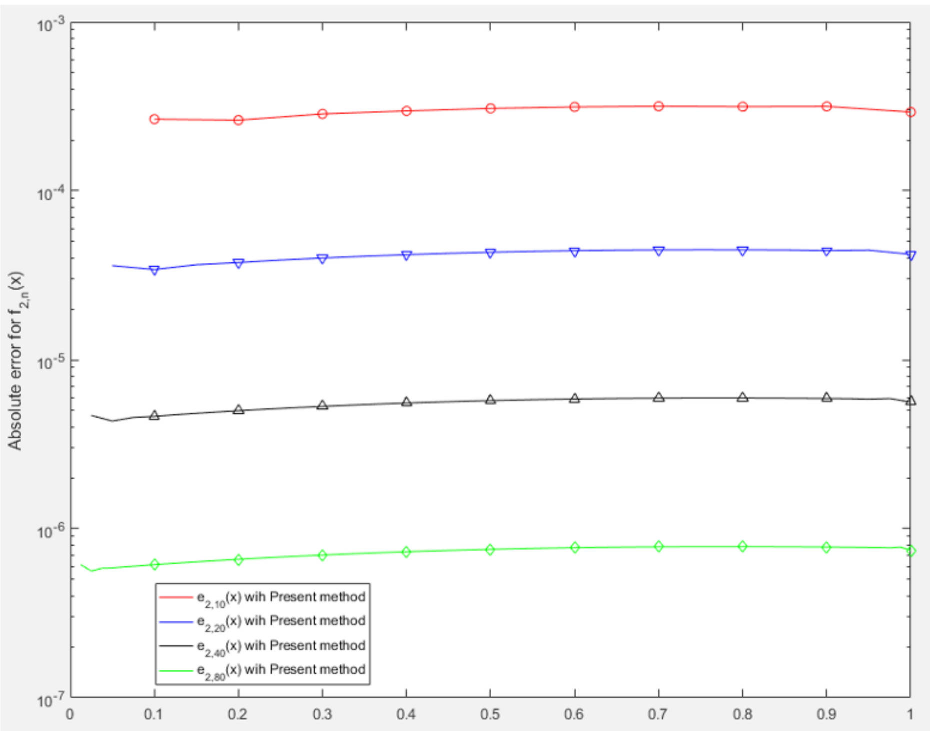

Fig. 6-2: Error comparison of function of $f_2(x)$ in Example 6.3

the first time. A reproducing kernel direct space is cleverly built, the reproducing space are reasonably simple because the author avoid the time consuming Schmidt orthogonalization process, and the approximate solution we get is no less than the second-order convergence. In the sixth section: Numerical examples, we do three experiments with the new algorithm, and make a comparison with other algorithms. In fact, this technique can be extended to other class of integral and differential equations. Although we just considered linear Volterra integral equations in our presentation, by that analogy, the algorithm can also be applied to systems of linear Fredholm-Volterra integral and Fredholm-Volterra integro-differential equations. From the illustrative tables and figures, we obtain that the algorithm is remarkably accurate and effective as expected.

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