Book Chapter

Continuous Wavelet Transform of Schwartz Distributions in $\mathcal{D}'_{L^2}(\mathbb{R}^n)$, n ≥ 1

Jagdish N Pandey

School of Mathematics and Statistics, Carleton University, Canada

***Corresponding Author:** Jagdish N Pandey, School of Mathematics and Statistics, Carleton University, Ottawa, ON K1S 5B6, Canada

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Abstract: We define a testing function space $D_{L^2}(\mathbb{R}^n)$ consisting of a class of C^∞ functions defined on $\mathbb{R}^n$, $n \geq 1$ whose every derivtive is $L^2(\mathbb{R}^n)$ integrable and equip it with a topology generated by a separating collection of seminorms $\{\gamma_k\}_{|k|=0}^{\infty}$ on $\mathcal{D}_{L^2}(\mathbb{R}^n)$, where $|k| = 0, 1, 2, \ldots$ and $\gamma_k(\phi) = \|\phi^{(k)}\|_2$, $\phi \in \mathcal{D}_{L^2}(\mathbb{R}^n)$. We then extend the continuous wavelet transform to distributions in $\mathcal{D}'_{L^2}(\mathbb{R}^n)$, $n \geq 1$ and derive the corresponding wavelet inversion formula interpreting convergence in the weak distributional sense. The kernel of our wavelet transform is defined by an element $\psi(x)$ of $\mathcal{D}_{L^2}(\mathbb{R}^n) \cap \mathcal{D}_{L^1}(\mathbb{R}^n)$, $n \geq 1$ which, when integrated along each of the real axes $X_1, X_2, \ldots X_n$ vanishes, but none of its moments $\int_{\mathbb{R}^n} x^m \psi(x) dx$ is zero; here $x^m = x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}$, $dx = dx_1\, dx_2 \cdots dx_n$ and $m = (m_1, m_2, \ldots m_n)$ and each of $m_1, m_2, \ldots m_n$ is ≥ 1. The set of such wavelets will be denoted by $\mathcal{D}_M(\mathbb{R}^n)$.

Keywords: wavelet transform; continuous wavelet transform; window functions; integral transform of generalized functions; Schwartz distributions

MSC: Primary: 42C40; 46F12; Secondary: 46F05; 46F10

1. Introduction

(a) Wavelet analysis has now entered into almost every walk of human life [1–5]. There are applications of wavelets in areas such as audio compression, communication, de-noising, differential equations, ECG compression, FBI fingerprinting, image compression, radar, speech and video compression, approximation, and so on. Discrete wavelet transform has many applications in Engineering and Mathematical sciences. Most notably, it is used for signal coding. Continuous wavelet transform is used in image processing. It is an excellent tool for mapping the changing properties of non-stationary signals.

Another recent example of an interesting application of wavelets was the LIGO experiment that detected gravitational waves using wavelets for signal analysis. See the paper submitted on 11 February 2016 to "General Relativity and Quantum Cosmology" [6].

These types of works on wavelets sparked the research on continuous wavelet transform of functions and generalized functions. The generalized function space that we chose for this work is the space $\mathcal{D}'_{L^2}(R^n)$ [7,8].

My earlier work in this connection was on the generalized function space $\mathcal{D}'(R^n)$, $n \geq 1$. The disadvantage in this space was that two functions having the same wavelet transform may differ by a constant even though all the moments of the wavelets are non-zero. The space $\mathcal{D}'_{L^2}(R^n)$, $n \geq 1$ does not contain a non-zero constant so that kind of difficulty is not encountered with this space. Besides, the space $\mathcal{D}'_{L^2}(R^n)$ is different from the generalized function space $\mathcal{D}'(R^n)$, $n \geq 1$.

(b) The wavelet that we will be dealing with is a variation of one dimensional wavelet xe^{-x^2}. All the even order moments of this wavelet are zero and so two functions having the same wavelet transform may differ by a polynomial. The kernel of the wavelet transform is generated by this wavelet with the formula $\frac{1}{\sqrt{|a|}}\psi\left(\frac{x-b}{a}\right)$, a, b real and $a \neq 0$ where $\psi(x) = xe^{-x^2}$. In order to remove the above mentioned problem we construct a wavelet

$(1+x-2x^2)e^{-x^2}$ such that all the moments $\int_{-\infty}^{\infty} x^m(1+x-2x^2)e^{-x^2}dx \neq 0$, $m \geq 1$ and $\int_{-\infty}^{\infty}(1+x-2x^2)e^{-x^2}dx = 0$. Many other wavelets satisfying this condition can be generated and some examples will be given in the coming section. An interesting point is that this wavelet is the union of a symmetric and an anti-symmetric wavelet as follows $(1+x-2x^2)e^{-x^2} = xe^{-x^2} + (1-2x^2)e^{-x^2}$. The wavelet xe^{-x^2} is antisymmetric and the wavelet $(1-2x^2)e^{-x^2}$ is symmetric and therefore this paper is very well suited to the journal "Symmetry".

(c) In the foregoing definition of the wavelet transform the kernel of the wavelet transform is

$$\frac{1}{\sqrt{|a|}}\psi\Big(\frac{x-b}{a}\Big),$$

where $\psi(x) = xe^{-x^2}$. We generalize the kernel ψ to dimensions $n > 1$ as $\psi(x) = x_1x_2\ldots x_ne^{-|x|^2}$ and the corresponding kernel of the wavelet transform $\frac{1}{\sqrt{|a|}}\psi\big(\frac{x-b}{a}\big)$. Clearly $\psi(x) \in \mathcal{D}_{L^2}(\mathbb{R}^n) \cap \mathcal{D}_{L^1}(\mathbb{R}^n)$. Therefore, two functions having the same wvelet transform may differ by a polynomial. We now illustrate this fact as follows. Let

$$\begin{aligned} f_1 &= f \\ f_2 &= f + \Big[\Big(\frac{t-b}{a}\Big)^m\Big]. \end{aligned}$$

These two distributions have the same wavelet transform but they may differ by a polynomial involving a constant term. See the calculation below:

$$\Big\langle\Big(\frac{t-b}{a}\Big)^m, \psi\Big(\frac{t-b}{a}\Big)\Big\rangle = \int_{\mathbb{R}^n}\Big(\frac{t-b}{a}\Big)^m\overline{\psi}\Big(\frac{t-b}{a}\Big)dt.$$

Put

$$\begin{aligned} x = \frac{t-b}{a} &= \int_{\mathbb{R}^n}|a|x^m\overline{\psi}(x)dx, \quad |a| = |a_1a_2\ldots a_n| \\ &= 0, \end{aligned}$$

when at least one of components $m_1, m_2 \ldots m_n$ is even and is $\neq 0$ when each of $m_1, m_2, \ldots m_n$ is odd.

So f_1 and f_2 will differ by the polynomial $\big(\frac{t-b}{a}\big)^m$. Therefore, in order that the uniqueness theorems may hold for the inversion formula for this wavelet transform, we have to select the kernel of our wavelet transform such that none of its moments of order m is zero, $m = (m_1, m_2, \ldots m_n)$, $m_i \geq 1\ \forall\ i = 1, 2, \ldots n$.

The dual of $\mathcal{D}(\mathbb{R}^n)$ contains a non-zero constant. The wavelet kernel that we are choosing belongs to $\mathcal{D}_{L^1}(\mathbb{R}^n) \cap \mathcal{D}_{L^2}(\mathbb{R}^n)$; as for example $x_1x_2\ldots x_n\phi(x)$ belongs to this space.

The kernel $\psi(x)$ of our wavelet transform should be such that $\int_{-\infty}^{\infty}\psi(x)dx_i = 0$, $i = 1, 2, \ldots n$ but none of its moments of order $m = (m_1, m_2, \ldots m_n)$ where each of $m_1, m_2, \ldots m_n$ is ≥ 1, is zero. If we take $\psi(x) = x_1x_2\ldots x_n\phi(x) \in \mathcal{D}(\mathbb{R}^n)$ then $\int_{-\infty}^{\infty}\psi(x)dx_i = 0$ but all its moments of order m, ie. $\int_{R^n} x^m\psi(x)dx$ will not be non-zero. We therefore seek our kernel $\psi(x) \in \mathcal{D}_{L^2}(\mathbb{R}^n)$ such that

$$\int_{-\infty}^{\infty}\psi(x)dx_i = 0, \quad i = 1, 2, 3, \ldots n$$

and

$$\int_{\mathbb{R}^n} x^m\psi(x)dx \neq 0, \quad m = (m_1, m_2, \ldots m_n).$$

In Section 3 we will show how such functions are selected or constructed.

2. Background Results

It is assumed that the readers are familiar with the elementary theory of distributions. Details of the theory may be found in [9–17].

A function $f \in L^2(\mathbb{R}^n)$ is called a *window function* [18–20] if $x_i f(x), \underset{i\neq j}{x_i x_j f(x)}, \underset{i\neq j\neq k\neq i}{x_i x_j x_k f(x)}$ $\ldots x_1 x_2 \ldots x_n f(x)$ belong to $L^2(\mathbb{R}^n)$, $i, j, k \ldots$ all assume values $1, 2, 3, \ldots n$. It is known that such a window function also belongs to $L^1(\mathbb{R}^n)$. A function $f \in L^2(\mathbb{R}^n)$ is said to be a *basic wavelet* if it satisfies the admissibility condition

$$\int_{\Lambda^n} \frac{|\hat{f}(\Lambda)|^2}{|\Lambda|} d\Lambda \qquad \text{is bounded,} \tag{1}$$

where $\hat{f}(\Lambda)$ is the Fourier transform of $f(x)$ defined by

$$\begin{aligned} \hat{f}(\Lambda) &= \lim_{N_1, N_2 \ldots N_n \to \infty} \frac{1}{(2\pi)^{n/2}} \int_{-N_n}^{N_n} \int_{-N_{n-1}}^{N_{n-1}} \cdots \int_{-N_2}^{N_2} \int_{-N_1}^{N_1} f(t) e^{-i\Lambda \cdot t} dt \\ &= \frac{1}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) e^{-i\Lambda \cdot t} dt \end{aligned}$$

$\Lambda = (\lambda_1, \lambda_2, \ldots \lambda_n)$, $t = (t_1, t_2 \ldots t_n)$, $dt = dt_1 dt_2 \ldots dt_n$ and the limit is interpreted in the $L^2(\mathbb{R}^n)$ sense ([21], p. 75).

In $L^2(\mathbb{R}^n)$, let us take $f(x) = x_1 x_2 \cdots x_n e^{-(x_1^2 + x_2^2 + \ldots + x_n^2)}$, then

$$\hat{f}(\Lambda) = \frac{\lambda_1 \lambda_2 \cdots \lambda_n i^n}{2^{3n/2}} e^{-(\lambda_1 + \lambda_2 + \cdots + \lambda_n)/4}.$$

Therefore

$$\int_{\mathbb{R}^n} \frac{|\hat{f}(\Lambda)|^2}{|\Lambda|} d\Lambda = \frac{1}{4^n} < \infty.$$

So $f(x)$ is a basic wavelet in $\mathbb{R}^n$.

We now describe some results proved in [20] which will be used in the sequel. These results are being stated for the convenience of our readers.

Theorem 1. *Let $f \in L^2(\mathbb{R}^n)$ be a window function on $\mathbb{R}^n$. Then $f \in L^1(\mathbb{R}^n)$ ([19], Theorem 3.1).*

Theorem 2. *Let $f : \mathbb{R}^n \to C$ be a $L^2(\mathbb{R}^n)$ window function. Let $f(\lambda_1, \lambda_2, \ldots, \lambda_n)$ be the Fourier transform of f defined by*

$$\begin{aligned} &\hat{f}(\lambda_1, \lambda_2, \ldots, \lambda_n) \\ = &\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} (f(x_1, x_2 \ldots x_n) e^{-i(x_1\lambda_1 + x_2\lambda_2 + \cdots + x_n\lambda_n)} dx_1 dx_2 \ldots dx_n \end{aligned}$$

Then the following statements are equivalent

(a) $\hat{f}(\lambda_1, \lambda_2, \ldots, \lambda_n)\|_{\lambda_j = 0} = 0$

(b) $\int_{-\infty}^{\infty} f(x_1, x_2, \ldots, x_j, \ldots, x_n) dx_j = 0,\ j = 1, 2, 3 \ldots n$ *([19], Theorem 3.2).*

Theorem 3. *Let $f \in L^2(\mathbb{R}^n)$ be a window function. Assume also that*

$$\int_{-\infty}^{\infty} f(x_1, x_2, \ldots x_i, \ldots x_n) dx_i = 0 \quad \forall i = 1, 2, \ldots, n.$$

Then, f satisfies the admissibility condition

$$\int_{\Lambda^n} \frac{|f(\lambda_1, \lambda_2 \cdots \lambda_n)|^2}{|\lambda_1 \lambda_2 \cdots \lambda_n|} d\lambda_1 d\lambda_2, \cdots d\lambda_n < \infty.$$

([19], Theorem 3.3).

More precisely we have

$$\begin{aligned} &\int_{\Lambda^n} \frac{|\hat{f}(\lambda_1, \lambda_2, \ldots, \lambda_n)|^2}{|\lambda_1 \lambda_2, \ldots, \lambda_n|} d\lambda_1 d\lambda_2 \cdots d\lambda_n \\ &\leq \Bigg[\|f\|_2^2 + 2^1 \sum_{i=1}^{n} ||x_i f||_2^2 + 2^2 \sum_{i,j=1, i\neq j}^{n} ||x_i x_j f||_2^2 \\ &\quad + 2^3 \sum_{i,j,k=1}^{n} ||x_i x_j x_k f||^2 + \cdots + 2^n ||x_1 x_2 \cdots x_n f||_2^2 \Bigg]. \end{aligned}$$

Theorem 4. *Let $f \in L^2(\mathbb{R}^n)$ be a window function. Then f satisfies the admissibility condition if and only if $\int_{-\infty}^{\infty} f(x_1, x_2, \ldots, x_n) dx_i = 0$, $i = 1, 2, 3, \ldots, n$.*

This is a corollary to the previous results.

Theorem 5. *Let $\phi \in \mathcal{D}_{L^2}(\mathbb{R}^n) \cap \mathcal{D}_{L^1}(\mathbb{R}^n)$; then ϕ satisfies the admissibility condition if and only if*

$$\int_{-\infty}^{\infty} \phi(x_1, x_2, \ldots, x_i, \ldots, x_n) dx_i = 0 \;\; \forall i = 1, 2, \ldots, n.$$

Now let us define a function $\psi(x)$ as follows

$$\psi(x) = x_1 x_2 \ldots x_n e^{-|x|^2}, \quad |x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

Clearly $\psi \in L^1(\mathbb{R}^n)$.

Then $\psi(x)$ is a window function belonging to $L^2(\mathbb{R}^n)$ and satisfying

$$\int_{-\infty}^{\infty} \psi(x) dx_i = 0 \quad \forall i = 1, 2, \ldots n.$$

Therefore in view of the foregoing results $\psi(x)$ is a wavelet.

Therefore, we define the wavelet transform of $f \in \mathcal{D}'_{L^2}(\mathbb{R}^n)$ by

$$\begin{aligned} W_f(a, b) &= \left\langle f(t), \frac{1}{\sqrt{|a|}} \psi\left(\frac{t-b}{a}\right) \right\rangle, \;\; a_i \neq 0, \; i = 1, 2, \ldots \\ &= 0, \;\; \text{when } a_i = 0, \text{ for any } i = 1, 2, \ldots n. \end{aligned}$$

Here

$$a = (a_1, a_2, \ldots a_n), \; b = (b_1, b_2, \ldots b_n).$$

$$\sqrt{|a|} = \sqrt{|a_1 \, a_2 \, \ldots \, a_n|}$$

$$\psi\left(\frac{t-b}{a}\right) = \psi\left(\frac{t_1 - b_1}{a_1}, \frac{t_2 - b_2}{a_2}, \ldots \frac{t_n - b_n}{a_n}\right).$$

(c) In the foregoing definition of the wavelet transform the kernel of the wavelet transform is

$$\frac{1}{\sqrt{|a|}} \psi\left(\frac{x-b}{a}\right),$$

where $\psi(x) = xe^{-x^2}$. We generalize the kernel ψ to dimensions $n > 1$ as $\psi(x) = x_1 x_2 \dots x_n e^{-|x|^2}$ and the corresponding kernel of the wavelet transform as $\frac{1}{\sqrt{|a|}}\psi(\frac{x-b}{a})$. Clearly $\psi(x) \in L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$. Therefore, two functions having the same wavelet transform may differ by a polynomial. We now illustrate this fact as follows. Let

$$f_1 = f$$

$$f_2 = f + \Big[\Big(\frac{t-b}{a}\Big)^m\Big].$$

These two distributions have the same wavelet transform but they differ by a polynomial involving a constant term. See the calculation below:

$$\Big\langle \Big(\frac{t-b}{a}\Big)^m, \psi\Big(\frac{t-b}{a}\Big)\Big\rangle = \int_{\mathbb{R}^n} \Big(\frac{t-b}{a}\Big)^m \psi\Big(\frac{t-b}{a}\Big) dt.$$

Put

$$\begin{aligned} x = \frac{t-b}{a} \quad &= \int_{\mathbb{R}^n} |a| x^m \psi(x) dx, \;\; |a| = |a_1 a_2 \dots a_n| \\ &= 0, \end{aligned}$$

when at least one of components $m_1, m_2, \dots m_n$ is even and is $\neq 0$ when each of $m_1, m_2, \dots m_n$ is odd.

So f_1 and f_2 will differ by the polynomial $\left(\frac{t-b}{a}\right)^m$. Therefore, in order that the uniqueness theorems may hold for the inversion formula for this wavelet transform, we have to select the kernel of our wavelet transform such that none of its moments of order m is zero, $m = (m_1, m_2, \dots m_n)$, $m_i \geq 1 \; \forall \; i = 1, 2, \dots n$.

The dual of $\mathcal{D}_{L^2}(\mathbb{R}^n)$ does not contain a non-zero constant. The wavelet kernel that we are choosing belongs to $D_{L^2}(\mathbb{R}^n) \cap \mathcal{D}_{L^1}(\mathbb{R}^n)$; as for example $x_1 x_2 \cdots x_n \, e^{-|x|^2}$ belongs to this space, but $\prod_{i=1}^{n} \Big(\frac{x_i^2}{\sqrt{1+x_i^6}}\Big)$ does not belong to the space $\mathcal{D}_{L^2}(\mathbb{R}^n) \cap D_{L^1}(\mathbb{R}^n)$.

The kernel $\psi(x)$ of our wavelet transform should be such that $\int_{-\infty}^{\infty} \psi(x) dx_i = 0$, $i = 1, 2, \dots n$ but none of its moments of order $m = (m_1, m_2, \dots m_n)$ where each of $m_1, m_2, \dots m_n$ is ≥ 1, is zero. If we take $\psi(x) = x_1 \, x_2 \dots x_n e^{-|x|^2}$ then $\int_{-\infty}^{\infty}(x) dx_i = 0$ but all its moments of order m, i.e., $\int_{R^n} x^m \psi(x) dx$ will not be non-zero. We therefore seek our kernel $\psi(x) \in \mathcal{D}_{L^2}(\mathbb{R}^n) \cap \mathcal{D}_{L^1}(\mathbb{R}^n)$ such that

$$\int_{-\infty}^{\infty} \psi(x) dx_i = 0, \quad i = 1, 2, 3, \dots n.$$

and

$$\int_{\mathbb{R}^n} x^m \psi(x) dx \neq 0, \quad m = (m_1, m_2, \dots m_n)$$

and $m_i \geq 1$, $\; i = 1, 2, \dots n$.

In Section 3 we will show how such a wavelet kernel is constructed.

3. Construction of Functions in the Space $\mathcal{D}_{L^2}(\mathbb{R}^n)$ Which Is a Wavelet Such That $\int_{\mathbb{R}^n} \phi(x) x^m dx \neq 0$, $m = (m_1, m_2, \cdots m_n)$; each $m_i \geq 1$, $i = 1, 2, \dots n$

I.e., Construction of functions $\psi(x) \in \mathcal{D}_M(R^n)$, $n \geq 1$.

In dimension $n = 1$ one such function is given as:

$$\phi(x) = (1 + x - kx^2) e^{-x^2}.$$

The constant k is so selected that

$$\int_{-\infty}^{\infty} \phi(x)dx = 0.$$

Therefore,

$$k = \frac{\int_{-\infty}^{\infty} e^{-x^2}dx}{\int_{-\infty}^{\infty} x^2 e^{-x^2}dx} = 2 = k_0.$$

A somewhat trivial construction of such a function $\phi(x)$ in n dimension can be a function $\psi(x)$ given by

$$\psi(x) = \prod_{i=1}^{n}(1 + x_i - 2x_i^2)e^{-x_i^2}.$$

One can see that

$$\int_{-\infty}^{\infty} \psi(x)dx_i = 0 \quad \forall i = 1, 2, \dots n$$

and

$$\int_{\mathbb{R}^n} \psi(x)x^m dx \neq 0.$$

for $m_1, m_2, \dots m_n \geq 1, m = (m_1, m_2, \cdots m_n)$.

We now give a non-trivial construction of such a wavelet as follows:

$n = 2$

$$\psi(x) = e^{-(x_1^2+x_2^2)} \left[1 + x_1 + x_2 + x_1x_2 - k_0(x_1^2 + x_2^2)\right].$$

We select the same constant k_0 as before, i.e., $k_0 = 2$. Clearly, integration along X_1 and X_2 respectively gives

$$\int_{-\infty}^{\infty} \psi(x)dx_i = 0, \quad i = 1, 2.$$

Verification of the fact that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi(x)x_1^{m_1}x_2^{m_2}dx_1dx_2;\ m_1 \geq 1,\ m_2 \geq 1$$

is non-zero is easy and is done as follows:

(i) $m_1,\ m_2$ both even,

$$\begin{aligned}&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi(x)x_1^{m_1}x_2^{m_2}dx_1dx_2\\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-|x|^2}\left[x_1^{m_1}x_2^{m_2} - k_0(x_1^2 + x_2^2)x_1^{m_1}x_2^{m_2}\right]dx \neq 0.\end{aligned}$$

(ii) $m_1,\ m_2$ both odd:

$$\begin{aligned}&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi(x)x_1^{m_1}x_2^{m_2}dx_1dx_2\\ = &\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x_1^2+x_2^2)}x_1^{m_1+1}x_2^{m_2+1}dx_1dx_2 > 0.\end{aligned}$$

(iii) m_1 even and m_2 is odd

$$\int_{R^2} \psi(x)\, x_1^{m_1}x_2^{m_2}dx_1dx_2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x_1^2+x_2^2)}\, x_1^{m_1}x_2^{m_2+1}dx_1dx_2 > 0.$$

when m_1 odd and m_2 even

$$\int_{R^2} \psi(x)\, x_1^{m_1}x_2^{m_2}dx_1dx_2 = \int_{R^2} e^{-(x_1^2+x_2^2)}\, x_1^{m_2}x_2^{m_1+1}dx_1dx_2 > 0.$$

$n = 3$. We then define $\psi(x)$ as

$$\psi(x) = e^{-(x_1^2+x_2^2+x_3^2)}\Big[1 + x_1 + x_2 + x_3 + x_1x_2 + x_2x_3 + x_3x_1 + x_1x_2x_3 - k_0(x_1^2 + x_2^2 + x_3^2)\Big].$$

The integral of $\psi(x)$ with respect to x_1, x_2, x_3 along the axes X_1, X_2, X_3 respectively is zero.

$$\int_{R^3} \psi(x)x^m dx \neq 0 \quad \text{for each} \quad m_1,\ m_2,\ m_3 \geq 1.$$

This fact can be verified similarly as in the case $n = 2$. Proceeding this way, a non-trivial construction of the function $\psi(x)$ can be done in any dimension $n > 1$.

4. Main Theorem

We hereby quote a theorem proved in ([11], p. 51, [17], p. 137) which plays a crucial role in the proof of our main theorem.

Theorem 6. *Let $f \in \mathcal{D}'(\mathbb{R}^n)$. We can find a sequence $\{f_k(x)\}_{k=1}^{\infty}$ of functions in $\mathcal{D}(\mathbb{R}^n)$ such that*

$$\lim_{k\to\infty} \langle f_k(x), \phi(x)\rangle = \langle f, \phi\rangle, \ \ \forall\phi \in \mathcal{D}(\mathbb{R}^n).$$

This fact is expressed by saying that $\mathcal{D}(\mathbb{R}^n)$ is dense in $\mathcal{D}'(\mathbb{R}^n)$. If is well known that by identifying $\phi(x) \in \mathcal{D}$ as a regular distribution, $\mathcal{D}(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n)$, ([11], p. 51, [17], p. 137).

Corollary 1 (Corollary to Theorem 6). *Let $f \in \mathcal{D}'_{L^2}(\mathbb{R}^n)$. We can find a sequence $\{f_k(x)\}_{k=1}^{\infty}$ of functions in $\mathcal{D}(\mathbb{R}^n)$ such that*

$$\lim_{k\to\infty} \langle f_k(x), \phi(x)\rangle = \langle f(x), \phi(x)\rangle \ \forall\, \phi \in \mathcal{D}(\mathbb{R}^n).$$

This fact is expressed by saying that $\mathcal{D}(\mathbb{R}^n)$ is dense in $\mathcal{D}'(\mathbb{R}^n)$ and so in $\mathcal{D}'_{L^2}(\mathbb{R}^n)$ as $\mathcal{D}'_{L^2}(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n)$ with identification similar to given in Theorem 6.

Lemma 1. *Let ψ be a wavelet belonging to the space $\mathcal{D}_{L^2}(\mathbb{R}^n) \cap \mathcal{D}_{L^1}(\mathbb{R}^n)$, $n \geq 1$ and $f \in \mathcal{D}'_{L^2}(\mathbb{R}^n)$ then the wavelet transform $F(a, b)$ of the distribution f with respect to wavelet function $\psi\left(\frac{x-b}{a}\right)$ is defined by*

$$F(a,b) = \left\langle f(x),\ \frac{1}{\sqrt{|a|}}\, \psi\left(\frac{x-b}{a}\right)\right\rangle,$$

$$x,\ a,\ b \in \mathbb{R}^n,\ a \neq 0.\ \textit{i.e., each}\ |a_i| > 0,\ i = 1, 2, \cdots n.$$

We wish to prove that $F(a, b)$ is a continuous function of a, b ; $a, b \in \mathbb{R}^n$.

Proof. We can write $F(a, b) = \frac{1}{\sqrt{|a|}} \left\langle f(x),\ \psi(\frac{x-b}{a})\right\rangle$ so it is enough to prove the continuity of $\langle f(x),\ \psi(\frac{x-b}{a})\rangle = G(a, b)$ (say)

$$G(a + \Delta a,\ b + \Delta b) - G(a, b) = \left\langle f(x),\ \psi\left(\frac{x - (b + \Delta b)}{a + \Delta a}\right) - \psi\left(\frac{x-b}{a}\right)\right\rangle.$$

So we need only show that $\psi\left(\frac{x-(b+\Delta b)}{a+\Delta a}\right) - \psi(\frac{x-b}{a}) \to 0$ in the topology of $\mathcal{D}_{L^2}(\mathbb{R}^n)$ as $\Delta a, \Delta b \to 0$ independently of each other. Now

$$D_x^k\left[\psi\left(\frac{x - (b + \Delta b)}{a + \Delta a}\right) - \psi\left(\frac{x-b}{a}\right)\right] = I \quad \text{(say)}.$$

Therefore

$$I = \left[\frac{1}{(a + \Delta a)^k}\, \psi^{(k)}\left(\frac{x - (b + \Delta b)}{(a + \Delta a}\right) - \frac{1}{a^k}\psi^{(k)}\left(\frac{x-b}{a}\right)\right].$$

Note that $\frac{x-b}{a}$ is $\left(\frac{x_1-b_1}{a_1}, \frac{x_2-b_2}{a_2}, \cdots \frac{x_n-b_n}{a_n}\right)$ and a similar explanation for $\frac{x-(b+\Delta b)}{a+\Delta a}$.

$$\begin{aligned} x &= (x_1, x_2, \ldots x_n) \\ a &= (a_1, a_2, \ldots b_n) \\ b &= (b_1, b_2, \ldots b_n) \\ k &= (k_1, k_2, \ldots k_n). \end{aligned}$$

In view of the mean value theorem of differential calculus of n-variables there exists a number $0 < \theta < 1$ such that ([22], p. 483)

$$\begin{aligned} I &= \sum_{i=1}^{n}\Bigg[-\frac{\psi^{(k+1)}}{(a+\theta\Delta a)^k}\left(\frac{x-(b+\theta\Delta b)}{a+\theta\Delta a}\right)\frac{x_i-(b_i+\theta\Delta b_i)}{(a_i+\theta\Delta a_i)^2}\Delta a_i \\ &\quad - \psi^k\left(\frac{x-(b+\theta\Delta b)}{a+\theta\Delta a}\right)\frac{k_i}{(a+\theta\Delta a)^k}\frac{1}{(a_i+\theta\Delta a_i)}\Delta a_i \\ &\quad - \frac{\psi^{(k+1)}}{(a+\theta\Delta a)^k}\left(\frac{x-(b+\theta\Delta b)}{a+\theta\Delta a}\right)\frac{\Delta b_i}{(a_i+\theta\Delta a_i)}\Bigg]. \end{aligned}$$

$\to 0$ in $\mathcal{D}_{L^2}(\mathbb{R}^n)$ as Δa and $\Delta b \to 0$ independently of each other. Convergence is with respect to x in the topology of $\mathcal{D}_{L^2}(\mathbb{R}^n)$. $\square$

The dual space $\mathcal{D}'_{L^2}(\mathbb{R}^n)$ of $\mathcal{D}_{L^2}(\mathbb{R}^n)$ does not contain a non-zero constant, therefore we will not use the notation $\mathcal{D}'_{L^2_F}(\mathbb{R}^n)$; this notation will also mean the space $\mathcal{D}'_{L^2}(\mathbb{R}^n)$.

Our main theorem is stated and proved as follows.

Theorem 7. *Let $f \in \mathcal{D}'_{L^2}(\mathbb{R}^n)$ and ψ be a wavelet belonging to the space $\mathcal{D}_{L^2}(\mathbb{R}^n) \cap \mathcal{D}_{L^1}(\mathbb{R}^n)$ then the wavelet transform of $f \in \mathcal{D}'_{L^2}(\mathbb{R}^n)$ with respect to the wavelet kernel $\psi \in \mathcal{D}_{L^2}(\mathbb{R}^n) \cap \mathcal{D}_{L^1}(\mathbb{R}^n)$ is defined as*

$$W_f(a,b) = \left\langle f(t), \frac{1}{\sqrt{|a|}}\psi\left(\frac{t-b}{a}\right)\right\rangle$$

or

$$\begin{aligned} &W_f(a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n) \\ &= \left\langle f(t_1, t_2, \ldots, t_n), \frac{1}{\sqrt{|a_1 a_2 \cdots a_n|}}\psi\left(\frac{t_1-b_1}{a_1}, \frac{t_2-b_n}{a_2}, \ldots, \frac{t_n-b_n}{a_n}\right)\right\rangle \end{aligned}$$

a_i and b_i are real and $a_i \neq 0$ $\forall\, i = 1, 2, \ldots, n$. It is asumed that $\int_{-\infty}^{\infty}\psi(x)dx_i = 0$ $\forall\, i = 1, 2, \ldots n$ and $\int_{\mathbb{R}^n} x^m \phi(x)dx \neq 0$, for $m = (m_1, m_2, \ldots, m_n)$ and $m_i \geq 1, i = 1, 2, \ldots n$.

Then

$$\begin{aligned} &\left\langle \left(\int_{\eta}^{A} + \int_{-A}^{-\eta}\right)\int_{-B}^{B}\frac{1}{C_\psi}\left\langle f(t), \frac{\psi\left(\frac{t-b}{a}\right)}{\sqrt{|a|}}\right\rangle\psi\left(\frac{x-b}{a}\right)\frac{dbda}{a^2\sqrt{|a|}}, \phi(x)\right\rangle \\ &\longrightarrow \langle f(x), \phi(x)\rangle \text{ as } A, B \to \infty \text{ and } \eta \to 0+ \end{aligned} \quad (2)$$

Here, when we say $B \to \infty$, it means that all the components $B_1, B_2, \ldots B_n$ of $B \to \infty$ independently of each other and similar notation for $A \to \infty$ and $\eta \to 0+$ means that all the components $\eta_1, \eta_2, \ldots \eta_n$ of $\eta \to 0$ independently of each other.

In (2) $db = db_1 db_2 \ldots db_n$ and the integration is being performed with respect to variables $b_1, b_2, \ldots b_n$ with the corresponding limit terms being $\int_{-B_n}^{B_n} \cdots \int_{-B_2}^{B_2}\int_{-B_1}^{B_1}$ and $da = da_1 da_2 \ldots da_n$ and the integration is being performed with respect to variables $a_1, a_2, \ldots a_n$ with the corresponding limit terms being

$$\left(\int_{\eta_n}^{A_n}+\int_{-A_n}^{-\eta_n}\right)\cdots\left(\int_{\eta_2}^{A_2}+\int_{-A_2}^{-\eta_2}\right)\left(\int_{\eta_1}^{A_1}+\int_{-A_1}^{-\eta_1}\right).$$

Proof. Since $\mathcal{D}(\mathbb{R}^n)$ is dense in $\mathcal{D}'_{L^2}(\mathbb{R}^n)$ we can find a sequence $\{f_n(t)\}_{n=1}^{\infty}$ in $\mathcal{D}(\mathbb{R}^n)$ such that

$$\lim_{k\to\infty}\langle f_k(t),\phi(t)\rangle=\langle f(t),\phi(t)\rangle\quad\forall\phi\in\mathcal{D}_{L^2}(\mathbb{R}^n)$$

Now

$$\left\langle\left(\int_{\eta}^{A}+\int_{-A}^{-\eta}\right)\frac{1}{C_\psi}\int_{-B}^{B}\left\langle f_k(t),\psi\left(\frac{t-b}{a}\right)\right\rangle\psi\left(\frac{x-b}{a}\right)\frac{dbda}{a^2|a|},\phi(x)\right\rangle \tag{3}$$

$$C_\psi=(2\pi)^n\int_{\Lambda^n}\frac{|\hat{\psi}(\Lambda)|^2}{|\Lambda|}d\Lambda<\infty\qquad\text{(admissibility condition)}$$

$\psi(\lambda_1,\lambda_2,\ldots\lambda_n)=\hat{\psi}(\Lambda)$, the Fourier transform of a window function $\psi(x)\in L^2(\mathbb{R}^n)$.

$$=\left\langle f_k(t),\frac{1}{C_\psi}\left(\int_{\eta}^{A}+\int_{-A}^{-\eta}\right)\int_{-B}^{B}\int_{-C}^{C}\phi(x)\bar{\psi}\left(\frac{x-b}{a}\right)\psi\left(\frac{t-b}{a}\right)\frac{dxdbda}{a^2|a|}\right\rangle \tag{4}$$

[using Fubini's Theorem]

Here also $dx=dx_1dx_2\ldots dx_n$ and integration is being performed with respect to variables $x_1,x_2,\ldots x_n$ with the corresponding limit terms being $\int_{-C_n}^{C_n}\cdots\int_{-C_2}^{C_2}\int_{-C_1}^{C_1}$ $C=(C_1,C_2,\ldots C_n)$ and $C\to\infty$ means all the components $C_1,C_2\ldots C_n$ of $C\to\infty$ independently of each other. Now letting $\eta\to0+,A,B,C\to\infty$ we see that

$$\begin{aligned}&\left\langle f_k(t),(P)\tfrac{1}{C_\psi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi(x)\bar{\psi}\left(\tfrac{x-b}{a}\right)\psi\left(\tfrac{t-b}{a}\right)\tfrac{dxdbda}{a^2|a|}\right\rangle\\&\qquad=\langle f_k(t),\phi(t)\rangle\end{aligned} \tag{5}$$

([18], Theorem 4.2). Each of the integral sign $\int_{-\infty}^{\infty}$ means $\underbrace{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}}_{(n\text{ times})}$ and similar meaning to the integral sign $\int_{-\infty}^{\infty}$ from now on. Therefore, from (3) and (5) we get

$$\lim_{\substack{A,B\to\infty\\ \eta\to0+}}\left\langle\left(\frac{1}{C_\psi}\int_{\eta}^{A}+\int_{-A}^{-\eta}\right)\int_{-B}^{B}\left\langle f_k(t),\psi\left(\frac{t-b}{a}\right)\right\rangle\psi\left(\frac{x-b}{a}\right)\frac{dbda}{a^2|a|},\phi(x)\right\rangle \tag{6}$$

$$=\langle f_k(x),\phi(x)\rangle.$$

The integral in (6) converges to $f_k(x)$ in $L^2(\mathbb{R}^n)$, ([19], Theorem 4.2).

So

$$\begin{aligned}&\left\langle(P)\tfrac{1}{C_\psi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\langle f_k(t),\psi\left(\tfrac{t-b}{a}\right)\right\rangle\psi\left(\tfrac{x-b}{a}\right)\tfrac{dbda}{a^2|a|},\phi(x)\right\rangle\\&\qquad=\langle f_k(x),\phi(x)\rangle.\end{aligned} \tag{7}$$

The integral in (7) converges to $f_k(x)$ in $L^2(\mathbb{R}^n)$. Now letting $k\to\infty$ we get

$$\begin{aligned}&\left\langle(P)\tfrac{1}{C_\psi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\langle f(t),\tfrac{1}{\sqrt{|a|}}\psi\left(\tfrac{t-b}{a}\right)\right\rangle\psi\left(\tfrac{x-b}{a}\right)\tfrac{dbda}{a^2\sqrt{|a|}},\phi(x)\right\rangle\\&\qquad=\langle f(x),\phi(x)\rangle.\end{aligned} \tag{8}$$

([19], Theorem 4.2).

The L.H.S. expressions in (2) and (8) are meaningful in view of Lemma 1. □

5. Conclusions

In order to deal with the wavelet transform of elements of $\mathcal{D}'$ we have to find wavelet function $\psi(x)$ in $\mathcal{D}$ satisfying the condition $\int_{-\infty}^{\infty} \psi(x)dx = 0$. It turned out that with $\phi(x) = \begin{cases} e^{-\frac{1}{1-x^2}} & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$ we construct a function $\psi(x) = x\phi(x)$, [18]. Then $\int_{-\infty}^{\infty} \psi(x)dx = 0$ and so it is a wavelet in view of results given in Section 2. The corresponding wavelet transform kernel will be $\frac{1}{\sqrt{|a|}}\psi\left(\frac{x-b}{a}\right)$, a, b real and $a \neq 0$. All even order moments of $\psi(x)$ is zero so two functions having the same wavelet transform will differ by a polynomial plus a constant; therefore we construct wavelet $\phi(x)$ in $\mathcal{D}$ such that none of its moments of order ≥ 1 is zero. It turned out that one such wavelet is $(1 + x - 2x^2)\phi(x)$ and the result is generalized in n dimension $n > 1$, many other functions were constructed. The disadvantage with this wavelet was that two functions having the same wavelet transform could differ by a constant. Bearing with the fault in our technique we generalized this result to dimension $n > 1$ and corrected this fault by deleting all non-zero constants from the space $\mathcal{D}'(\mathbb{R}^n)$, $n \geq 1$ [18].

If we look into the generalized function space $\mathcal{D}'_{L^2}(\mathbb{R}^n)$ we find that this space does not contain a non-zero constant. For this reason, this space is quite interesting and using technique similar to that used for the space $\mathcal{D}'(R^n)$, $n \geq 1$ we construct wavelet function $\phi \in \mathcal{D}_{L^2}(\mathbb{R}^n)$ whose all moments of order $m = (m_1, m_2, \ldots m_n)$ each of $m_1, m_2, \ldots m_n$ is ≥ 1 are non-zero. We then proved the wavelet inversion formula for the space $\mathcal{D}'_{L^2}(\mathbb{R}^n)$, $n \geq 1$ using these results derived. Uniqueness theorem for the inversion formula then follows.

There are many applications of wavelets and continuous wavelet transforms which are mentioned in the beginning part of the introduction.

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References

1. Bogess, A.; Narcowich, F.J. *A First Course in Wavelets with Fourier Analysis*; Addison-Wesley: Upper Saddle River, NJ, USA, 1984.
2. Chui, C.K. An introduction to wavelets. In *Wavelet Analysis and Its Applications*; Academic Press, Inc.: Boston, MA, USA, 1992; Volume 1, pp. x+264.
3. Daubechies, I. Ten lectures on wavelets. In *CBMS-NSF Regional Conference Series in Applied Mathematics*; Society for Industrial and Applied Mathematics (SIAM): Philadelphia, PA, USA, 1992; Volume 16, pp. xx+357. [CrossRef]
4. Keinert, F. *Wavelets and Multiwavelets*; Studies in Advanced Mathematics; Chapman & Hall/CRC: Boca Raton, FL, USA, 2004; pp. xii+275.
5. Meyer, Y. Wavelets and operators. In *Cambridge Studies in Advanced Mathematics*; Cambridge University Press: Cambridge, UK, 1992; Volume 37, pp. xvi+224.
6. Abbott, B.P. Observation of Gravitational Waves from a Binary Black Hole Merger. *Phys. Rev. Lett.* **2016**, *116*, 06112. [CrossRef] [PubMed]
7. Pandey, J.N.; Jha, N.K.; Singh, O.P. The continuous wavelet transform in *n*-dimensions. *Int. J. Wavelets Multiresolut. Inf. Process.* **2016**, *14*, 1650037. [CrossRef]
8. Pandey, J.N.; Maurya, J.S.; Upadhyay, S.K.; Srivastava, H.M. Continuous wavelet transform of Schwartz tempered distribution in $S'(\mathbb{R}^n)$. *Symmetry* **2019**, *11*, 235. [CrossRef]
9. Gelfand, I.M.; Shilov, G.E. *Generalized Functions*; Spaces of Fundamental and Generalized Functions, Translated from the 1958 Russian Original [MR0106409] by Morris D. Friedman, Amiel Feinstein and Christian P. Peltzer, Reprint of the 1968 English Translation [MR0230128]; AMS Chelsea Publishing: Providence, RI, USA, 2016; Volume 1, pp. xviii+423. [CrossRef]
10. Gelfand, I.M.; Shilov, G.E. *Generalized Functions*; Spaces of Fundamental and Generalized Functions, Translated from the 1958 Russian Original [MR0106409] by Morris D. Friedman, Amiel Feinstein and Christian P. Peltzer, Reprint of the 1968 English Translation [MR0230128]; AMS Chelsea Publishing: Providence, RI, USA, 2016; Volume 2, pp. x+261. [CrossRef]