Book Chapter

Entropy Valuation of Option: Using Option-implied Information

Xisheng Yu*

School of Economic Mathematics, Southwestern University of Finance and Economics, China

***Corresponding Author:** Xisheng Yu, School of Economic Mathematics, Southwestern University of Finance and Economics, China

Published **April 08, 2021**

This Book Chapter is a republication of an article published by Xisheng Yu at Entropy in July 2020. (Yu, X. Risk-Neutrality of RND and Option Pricing within an Entropy Framework. Entropy 2020, 22, 836. https://doi.org/10.3390/e22080836)

How to cite this book chapter: Xisheng Yu. Entropy Valuation of Option: Using Option-implied Information. In: Ricardo Beltran-Chacon, editor. Entropy: Theory and New Insights. Hyderabad, India: Vide Leaf. 2021.

Abstract

This paper studies an entropy valuation method by incorporating informative risk-neutral moments recovered from options as constraints. Within this entropy framework, a unique distribution close enough to the correct one is obtained and its risk-neutrality is deeply verified based on simulations. Using this resultant risk-

neutral distribution, a sample of risk-neutral underlying price paths is generated and ultimately the option prices are computed. The pricing performance and analysis in simulations demonstrate that this proposed valuation is completely comparable to the benchmarks, and can produce amazingly accurate prices for options.

Keywords

Risk-Neutral Moment; Risk-Neutral Distribution; Entropy Valuation; Risk-Neutrality; Option Pricing

JEL Classification: G13

Introduction

The key issue in applying the risk-neutral pricing method to option pricing is to find an appropriate risk-neutral pricing measure (or risk-neutral probability distribution, RND). With absence of arbitrage, a perfect and complete market ensures the existence of a unique equivalent martingale measure (as the RND) [1,2]. In the realistic market system, however, it allows many candidates for the RND due to the incompleteness, and one has to choose one particular measure by using nonparametric or model-independent methods. Some retrospective studies on nonparametric valuations can be found in Hutchinson *et al.* (1994), [3-5].

Among those above, an entropy-based pricing method, Canonical valuation was initialized by Stutzer [6] for pricing European option. The idea is that between two densities/distributions[1] satisfying the martingale restriction, the one that is more ‘uncertain’ should be chosen. In general, applying the criterion of entropy delivers convincing results. For example, on the unit interval [0,1], if no constraints are given, the density with the maximum (greatest) entropy is the uniform density. Using the maximum entropy principle, the Canonical

[1] One is an empirical distribution governing the underlying asset’s return, and another one is its risk-neutral counterpart utilized to price the option.

method transforms the empirical distribution of the observed stock returns at maturity into a risk-neutral distribution (RND, also called maximum entropy distribution or *Canonical martingale measure*), which is then used to price option. With the martingale constraint, the resultant RND derived within Canonical valuation framework is a martingale measure and is reasonably regarded as the 'best' RND for option pricing. When deriving the risk-neutral measure, Canonical approach does not require any normative assumptions for the underlying dynamics but rather relies upon the available cross-sectional market-information, and for this reason, the approach is called model-free or nonparametric.

Owing to the appealing feature, this entropy-based Canonical valuation has been theoretically and empirically surveyed in the literature. Buchen and Kelly [7] examine the ability of pricing European options in an incomplete market by extracting the RND from option prices and further discusses a relative entropy principle for option pricing in a simulated market. Shortly after, Stutzer [8] mathematically shows that, with a sole martingale constraint, the RND (Canonical martingale measure) derived from his Canonical framework is equivalent to the Black-Scholes measure. Gray *et al.* [26] price index options and find that the Canonical measure incorporating a small amount of cross-sectional information outperforms the Black-Scholes model and generates more effective hedging ratios. To shrink the feasible set of measures more tightly around the correct martingale measure, Alcock and Auerswald [9] use more price-sensitive information as a second constraint by setting a specific call option to be correctly priced, to produce a more accurate option price than Stutzer's original canonical price. Neri and Schneider [10] also investigate the proposition of Buchen-Kelly's density within the family of entropy-maximizing densities and find that the densities converge to Buchen-Kelly density in the sense of relative entropy. Some recent researches related to entropy-valuation can also be found in Yu and Yang [11], Alcock and Smith [27], and more recently, Liu *et al.* [12] and Feunou and Okou [13].

Despite the attractiveness of *Canonical valuation* and the entropy-based extensions above, they could be improved when adding more option-implied informative constraints, into the corresponding entropy frame, rather than using the sole martingale condition or imposing a second constraint. Indeed, option prices contain much efficient information (such as the volatility smile and tail behavior) about market participants' perceptions of the distribution of the underlying return which accurately captures the shape of the correct RND (see e.g., [14-18]). From the perspective of statistics, for instance, a normal distribution can be identified using the first- and second-order moments. Hence, if one can retrieve the *risk-neutral moments*[2] *(RNMs)* from option prices and incorporate them as constraints into the entropy framework, the correct RND can be accurately estimated, and consequently produces a fairly accurate price for the being-priced-option.

Motivated by this, this paper sets up an entropy valuation framework by incorporating the informative risk-neutral moments (RNMs) serving as constraints, nesting the sole martingale constrait, into Stutzer's Canonical valuation, within which the 'best' risk-neutral distribution (RND) is derived as the equivalent martingale measure for pricing European options. To deeply assess the efficacy of this RNMs-based entropy valuation, the accuracy of exacted RNMs and the risk-neutralities of resultant RND are verified ultimately the pricing performance is fully evaluated. This proposed entropy valuation has the following advantages. *First*, the implementation of this method is relatively tractable due to that the method does not need to impose pre-assumptions on either the market structure or the dynamics of underlying asset price. Meanwhile, extracting the RNMs is also model-free. Further, the RNM constraints we used in the valuation framework nest the single martingale constraint in the original canonical valuation. *Second,* the informative RNMs serving as constraints can be estimated with a high accuracy using a small amount of option data, and in this way, we can learn much more about the shape of the correct RND since the option-contained information, such as volatility smile,

[2] That is, the moments under the risk-neutral probability distribution.

skewness and excess kurtosis, is exploited by the RNMs. *Consequently*, such obtained RND can be reasonably chosen as the 'best' one for option pricing, and produces a fairly precise price, as expected.

The implementation of our RNMs-based entropy pricing framework proceeds in three steps. *First*, the RNMs are recovered using market-available option prices and incorporated into the entropy framework as constraints, *then* the RND is derived as the pricing measure within the framework. *Ultimately* the price of option is calculated according to the risk-neutral pricing principle. Following these, we contribute to the literature step by step: *First*, simulation experiments are conducted to check whether the RNMs can be accurately estimated using option prices. *More importantly*, the risk-neutrality of derived RND is also experimentally verified. *Finally*, in the simulation setting, we present a proof that the values of RNMs indeed correspond to the true values, and as expected, the risk-neutrality of the resultant risk-neutral measure (RND) is also fully confirmed. We further demonstrate the pricing efficacy of the proposed method by benchmarking the estimated price against the true price. Fortunately, all the results of those above encourage us to argue that the RNMs-constrained entropy valuation framework we established offers an attractive choice for option pricing.

The remainder of this paper is organized as follows. The RNMs-based entropy valuation framework incorporating the informative RNMs is established and the pricing scheme is derived in Section 2. We present propositions of RNMs and RND and conduct simulation experiments to verify the correctness of estimated RNMs and the risk-neutrality of RND in Section 3. Section 4 provides the pricing result and analysis, and Section 5 ends with conclusions.

Entropy Valuation with RNM-Constraints

This section constructs the entropy valuation by imposing the RNM restrictions on this framework and provides the pricing scheme by deriving the RND. With the RNM constraints replacing the single martingale constraint, a more proper RND is derived as the risk-neutral measure for option pricing. Not that, again, such obtained RND can correctly reflect the asset price behavior such as volatility smile, skewness and excess kurtosis, so that the options can be accurately priced by using this RND.

Pricing Scheme

Starting with some notation frequently used throughout this paper. Assume the current time $t_0 = 0$ and the underlying asset pays dividends during the life of option with a yield q. We denote the price of the underlying asset at time t by S_t ($0 \le t \le T$) and the maturity date of the option by T. Let theτ-period underlying log-return at time t be given as the price relative $R_{t,\tau} = \ln(S_{t+\tau}/S_t)$, and$\tau$-period jth-order RNM at time t be defined as $m_{t,\tau}(j) = E^{\pi^Q}([R_{t,\tau}]^j)$, where the symbol $E^{\pi^Q}(\cdot)$ represents the expectation operator under the risk-neutral probability measure π^Q, andτcan be any appropriate time period such as the time to maturity or a day. A special case is where$\tau$is the time to maturity ($\tau = T - t_0$). As usual, we denote the time zero price of European option with strike K and maturity T by $C(T,K)$ for call and $P(T,K)$ for put.

According to Yu and Yang [18] and Bakshi *et al.* [28][3], the risk-neutral moments of gross return $m_{t,\tau}(j)(j = 1,2,...)$ can be expressed as an integral function of option prices as below.

Lemma 1 (RNM Representation) *Under the martingale pricing measure* π^Q*, the risk-neutral moment (RNM)* $m_{t,\tau}(j)$ *can be recovered from the market prices of out-of-the-money (OTM)*[4] *call and put options as follows.*

The $(T - t_0)$*-period first order RNM* $m_{t_0,\tau}(1)$ *is expressed as*

[3] Bakshi *et al.* [28] use a Taylor expansion to obtain the expressions for the first four RNMs, which has been extended to an arbitrage order *j*th-order RNMs by Yu and Yang [18] using the characteristic function instead.

[4] A call/put is out-of-the-money (OTM) if its strike price is above/below the current underlying price; at-the-money (ATM) if equal to underlying price and in-the-money (ITM) if below/above underlying price.

$$m_{t_0,\tau}(1) = e^{(r-q)(T-t_0)} - e^{r(T-t_0)}\left[\int_{S_0}^{\infty}\frac{1}{K^2}C(T,K)dK + \int_0^{S_0}\frac{1}{K^2}P(T,K)dK\right] - 1 \qquad (1)$$

and $(T-t_0)$ *-period jth-order RNM* $m_{t_0,\tau}(j)$ *(* $j \geq 2$ *) is given by*

$$m_{t_0,\tau}(j) = je^{r(T-t_0)} \times [\int_{S_0}^{\infty}\frac{(j-1)-\ln(K/S_0)}{K^2}(\ln(K/S_0))^{(j-2)}C(T,K)dK$$

$$+\int_0^{S_0}\frac{(j-1)-\ln(K/S_0)}{K^2}(-\ln(K/S_0))^{(j-2)}P(T,K)dK], \qquad (2)$$

where S_0 *denotes the current price of underlying asset, r is the continuously compounded risk-free interest rate matching the time to the option maturity and q the dividend yield*[5]*. Both r and q are annualized and assumed to be constant given the time to maturity.*

From Lemma 1, RNMs are expressed as the integrals of option prices over a range of strike prices $[0,S_0)$ and $[S_0,\infty)$ with two singular points 0 and ∞. We would discuss the calculations of RNMs later.

To construct the RNM-constrained entropy framework, we consider a sequence of historical price of underlying asset which is used to produces the $(T-t_0)$-period log-return series $R_i = \ln(S_{t_{-(i-1)}}/S_{t_{-(i-1)-T}})$ where t_{-i} is the time point prior to current time t_0 $(t_0 = 0)$ for all $i = 1,2,\ldots,N$. Suppose that no subjective knowledge is imposed in advance for the asset price/return dynamics, we therefore should assign each return R_i equal probabilities $\pi_i^P = 1/N$ as the prior empirical distribution. Hence the return process with such measure $\pi^P = (\pi_1^P,\ldots,\pi_N^P)$ is a sample estimate of the true asset return dynamics. Now the RNM-constrained entropy pricing scheme comes to the following problem:

[5] The case of discrete dividend payments can also be readily considered.

$$\begin{cases} \pi^Q = \underset{\pi_i^Q>0}{\arg\min} \sum_{i=1}^{N} \pi_i^Q log(\pi_i^Q / \pi_i^P) \\ s.t. \begin{cases} \sum_{i=1}^{N} \pi_i^Q R_i^j = m_{t_0,\tau}(j) \\ m_{t_0,\tau}(0) = 1 \end{cases}, j = 0,1,2,\ldots, J \end{cases} \tag{3}$$

where $\pi^Q = (\pi_1^Q, \ldots, \pi_N^Q)$ is the risk-neutral probability distribution (RND, also an equivalent martingale measure) of the underlying log-return we are seeking to price options, and as defined, R_i is the log-return observations and $m_{t_0,\tau}(j)$ is the RNMs of log-return with $t_0 = 0$ and $\tau = T$. Note that the RNM constraints subsume the single martingale constraint used in Stutzer's framework, since theoretically the first- and second-order RNM constraints imply the martingale constraint and variance constraint.

Calculations of RNM and Derivation of RND

Within this RNM-constrained entropy model (3), the pricing scheme is then derived by specifying the calculation of RNMs and the derivation of RND.

Calculations of RNM

Lemma 1 shows that the RNMs are the integrals of option prices over a range of strike prices $[0, S_0)$ and $[S_0, \infty)$ with two singular points 0 and ∞. Given a continuum of strike prices over these intervals, calculating the integrals via a numerical method is straightforward. However, only a finite number of traded options with discrete strike prices are available in a real market. Following the convention of some literature, we employ the trapezoidal numerical method to numerically evaluate the integral and use a practical and effective curve-fitting method to handle the issue of option availability. *The operational procedures are outlined as follows,* for details see Jiang and Tian [17] and Yu and Yang [18].

First, the intervals of integration $[0, S_0)$ and $[S_0, +\infty)$ are split into three subintervals, $[0, S_0) = [0, K_0) \cup [K_0, K_{min}) \cup [K_{min}, S_0)$ and

$[S_0,+\infty)=[S_0,K_{max})\bigcup[K_{max},K_\infty)\bigcup[K_\infty,+\infty)$, respectively, where K_{min} is the minimum available strike price and K_{max} the maximum one, whereas K_0/K_∞ is a very small/large number (in this study setting $K_0=0.2K_{min}$, $K_\infty=5K_{max}$) so that a put/call option with strike prices in $[0,K_0)/[K_\infty,\infty)$ is valueless. The integrals over $[0,K_0)$ and $[K_\infty,\infty)$ in (1) and (2) are therefore zero. *Second,* the integrals over other intervals need to be calculated. As shown from the integrals, the required strike prices are beyond the range of the available data. The option prices corresponding to such strike prices need to be inferred from the given option prices. A curve-fitting method is adopted to handle this restriction by constructing a set of implied volatilities from observed option prices via Black-Scholes option formula[6]. *Third,* for the integrations over $[K_{min},S_0)$ and $[S_0,K_{max})$, we use two constants (endpoint implied volatilities) to extrapolate the option prices for these two intervals beyond the available range. The extrapolation is truncated at the strike points, denoted as K_0 and K_∞. *Fourth,* when approximating the integrals using a numerical integration method, two types of Riemann integral sum are utilized. Specifically, Riemann sums of the left endpoints, as well as the right endpoints are first calculated, and their average is then used as an approximation of the required integral. In this study, we adopt a trapezoidal numerical method and each of the intervals involved in integration is divided into a number m (e.g., $m = 80$) of equidistant subintervals.

Derivation of RND

Theorem 1 (RND Solution) *Assume the prior distribution is uniform* $\pi_i^P=1/N$, *and consider a time interval* $\tau=T$. *Then the optimal solution of equivalent martingale measure RND is obtained by solving the optimization problem (3),*

---

[6] First, implied volatilities are calculated using the Black-Scholes formula based on the selected set of option prices. Second a cubic spline function is used to interpolate the implied volatilities and infer implied volatilities at strike points located in $[S_0, K_{max}]$ or $[K_{min}, S_0]$ from the fitted function. Third, using again the Black-Scholes formula to inversely map the inferred volatilities so as to obtain the required option prices. Note that the Black-Scholes formula here is merely used as a tool to build a smooth nonlinear relation between volatility and option prices.

$$\hat{\pi}_i^Q = \frac{exp(\sum_{j=1}^{J} \lambda_j^* R_i^j)}{\sum_{i=1}^{N} exp(\sum_{j=1}^{J} \lambda_j^* R_i^j)} \tag{4}$$

where the Lagrange multiplier vector $\lambda^* = (\lambda_1^*, \lambda_2^*, \ldots \lambda_J^*)$ *is found numerically*[7] *by the following optimization,*

$$\lambda^* = \underset{\lambda_1, \ldots, \lambda_J}{arg\ min} \sum_{i=1}^{N} exp\left(\sum_{j=1}^{J} \lambda_j [R_i^j - m_{t_0,T}(j)]\right) \tag{5}$$

Proof: Here we provide a fairly simple and easily-understood way to finish this proof, although it can also be solved by employing a dual method, see Ben-Tal [19], etc..

Note that Problem (3) is a strictly convex optimization and has a unique global optimal solution. Then we employ Lagrange multiplier method for solution seeking. The Lagrangian function for the constrained optimization problem (3) is given by (note that $\pi_i^P = 1/N$),

$$L(\pi^Q; \lambda_0, \lambda) = \sum_{i=1}^{N} \pi_i^Q log(\pi_i^Q) + \lambda_0 (1 - \sum_{i=1}^{N} \pi_i^Q) + \sum_{j=1}^{J} \lambda_j [m_{t_0,T}(j) - \sum_{i=1}^{N} \pi_i^Q R_i^j]$$

Where $\lambda = (\lambda_1, \ldots, \lambda_J)$ is the Lagrange multiplier. Then the first-order conditions are

$$\frac{\partial L}{\partial \pi_i^Q} = 1 + log(\pi_i^Q) - \lambda_0 - \sum_{j=1}^{J} \lambda_j R_i^j = 0$$

which leads to

[7] Lagrange multiplier λ^* is calculated by first obtaining the optimal value λ_0 via quasi-Newton method as the initial value, and then running Nelder–Mead simplex search for the final optimal Lambda value. The reason for doing this is that, theoretically Nelder–Mead simplex search method is more stable, while the frequently used quasi-Newton method is faster. For more discussions, see Agmon et al. [29] and Mead and Papanicolaou [30].

$$\pi_i^Q = exp[\sum_{j=1}^{J} \lambda_j R_i^j - (1-\lambda_0)] \tag{6}$$

Summing these probabilities to one (i.e., the constraint $m_{t_0,\tau}(0)=1$) implies

$$\sum_{i=1}^{N} exp(\sum_{j=1}^{J} \lambda_j R_i^j) = exp(1-\lambda_0) \tag{7}$$

Substituting (7) into (6) yields the solution of π_i^Q, as desired,

$$\pi_i^Q = \frac{exp(\sum_{j=1}^{J} \lambda_j R_i^j)}{\sum_{i=1}^{N} exp(\sum_{j=1}^{J} \lambda_j R_i^j)} \tag{4'}$$

Applying formula (4') to constraint equations $\sum_{i=1}^{N} \pi_i^Q R_i^j = m_{t_0,T}(j)$ produces that,

$$\sum_{i=1}^{N} \frac{exp(\sum_{j=1}^{J} \lambda_j R_i^j) R_i^j}{\sum_{i=1}^{N} exp(\sum_{j=1}^{J} \lambda_j R_i^j)} = m_{t_0,T}(j)$$

equivalent to the following by rearranging,

$$\sum_{i=1}^{N} exp\left(\sum_{j=1}^{J} \lambda_j [R_i^j - m_{t_0,T}(j)]\right)\left(R_i^j - m_{t_0,T}(j)\right)\left(exp[\sum_{j=1}^{J} \lambda_j m_{t_0,T}(j)]\right) = 0$$

or, $\sum_{i=1}^{N} exp\left(\sum_{j=1}^{J} \lambda_j [R_i^j - m_{t_0,T}(j)]\right)\left(R_i^j - m_{t_0,T}(j)\right) = 0$

in which the left-hand side is exactly a partial derivative,

$$\frac{\partial}{\partial \lambda_j}\left(\sum_{i=1}^{N} exp\left(\sum_{j=1}^{J} \lambda_j [R_i^j - m_{t_0,T}(j)]\right)\right) = 0 \tag{8}$$

Now return to optimization problem (5), the derived equation (8) is the first condition of problem (5), and note that the objective function in (5) is strictly convex, therefore the Lagrange multiplier λ in (4') satisfying (8) must be the unique solution to problem (5).

Risk-Neutral Underlying Paths and Option Price

The resultant distribution $\hat{\pi}^Q = (\hat{\pi}_1^Q, \ldots, \hat{\pi}_N^Q)$ in (4) represents the occurrence of the empirical log-return $R_i = \ln(S_{t_{-(i-1)}} / S_{t_{-(i-1)-T}})$ and serves as the risk-neutral probability measure we need for option pricing. Now with the risk-neutral probability $\hat{\pi}^Q$, an independent random sample of future log-returns, $\tilde{R} = (\tilde{R}_1, \ldots, \tilde{R}_M)$, can be drawn from the set of historical log-returns $\{R_i\}_{i=1}^N$ by employing the inverse transform method [20], and then used to directly generate M risk-neutral price paths for the underlying asset at option maturity T,

$$S_T^{(k)} = S_0 \tilde{R}_k \quad (k = 1, 2, \ldots, M)$$

where $\tilde{R}_k$ is the kth random sample of the log-return corresponding to the kth underlying price path, and S_0 is the underlying price at initial time $t_0 = 0$.

Next, we calculate the European option price. Since the simulated sample of underlying price paths is under the risk-neutral measure $\hat{\pi}^Q$, as specified above, it is therefore risk-neutral. Hence according to risk-neutral pricing method, directly averaging the discounted payoff of all paths yields the final value of the option. Specifically, a call/put option expiring at time T with strike K, can be valued as the following,

The call price is given as,

$$C(T, K) = \frac{1}{M} \sum_{k=1}^{M} e^{-r(T - t_0)} (S_T^{(k)} - K)^+ \tag{9}$$

The put price is given as,

$$P(T, K) = \frac{1}{M} \sum_{k=1}^{M} e^{-r(T - t_0)} (K - S_T^{(k)})^+ \tag{10}$$

where r is the interest rate matching the time t_0 to T and a^+ takes the maximum of zero and a.

Verification of Correctness of Extracted RNMs and Risk-Neutrality of RND

Due to the significance of RNMs in deriving RND, it is indispensable to verify the correctness of calculated RNMs in Lemma 1, as well as the risk-neutrality of obtained RND in (4). To complete this verification, a market that provides a true value for each RNM and a real RND is needed, so that we can facilitate the comparisons between the estimated RNMs and the true RNMs, and between the derived RND and the real RND. Considering such requirements, we choose a simple but effective Black-Scholes (B-S) market in which the exact RNMs and RND can be easily derived.

Correctness of the Estimated RNMs

Within the B-S setting, a geometric Brownian motion (GBM) is assumed for the underlying process S_t,

$$\frac{dS_t}{S_t} = (\mu - q)dt + \sigma d\omega_t \tag{11}$$

where μ is the growth rate, q the continuous dividend yield and ω_t the standard Wiener process. The continuously compounded log-return is then normally distributed and given by,

$$R_{t_0,T} = (\mu - q - \frac{\sigma^2}{2})T + \sigma\sqrt{T}\varepsilon \tag{12}$$

where ε is standard normal.

According to the definition of RNM of log-return, the RNM in B-S world is then expressed as $m^{BS}_{t_0,T}(j) = E^{\pi^{BS}}([R_{t_0,T}]^j)$, where π^{BS} is the B-S risk-neutral measure. Conceptually, within a B-S market, the RNMs $m_{t_0,T}(j)$ (reminding that $m_{t_0,T}(j) = E^{\pi^Q}([R_{t_0,T}]^j)$) and the RND π^Q derived from our entropy valuation framework should be the same with $m_{t_0,T}^{BS}(j)$ and π^{BS}, respectively. Meanwhile it is notable that the B-S risk-neutral measure can be uniquely determined by the first two moments of log-return, $m_{t_0,T}^{BS}(1)$ and $m_{t_0,T}^{BS}(2)$, because the log-price is normally distributed following Eq. (12) and the risk-neutral distribution is exactly characterized by its first and second moments (μ-q) and volatilityσ. Consequently, in this context of B-S setting, we *choose to use two RNM constraints in our entropy framework (3) by taking J=2.*

The following Theorem 2 states the correctness of the estimated RNMs $m_{t_0,T}(j)$ in Lemma 1 by verifying the equivalence relation between RNMs $m_{t_0,T}(j)$ and B-S RNM $m_{t_0,T}^{BS}(j)$ (j=1,2).

Theorem 2 (Equivalence of RNM) *Within the B-S setting, denote the first two order moments of log-return under the B-S martingale measure by* $m_{t_0,T}^{BS}(j) = E^{\pi^{BS}}([R_{t_0,T}]^j)$ *(j=1,2), where* π^{BS} *is the B-S risk-neutral measure, then for the risk-neutral moments* $m_{t_0,T}(j)$ *using options detailed in Lemma 1, we have that:*

$$m_{t_0,T}(1) = m_{t_0,T}^{BS}(1) = (r - q - \frac{\sigma^2}{2})(T - t_0) \tag{13}$$

$$m_{t_0,T}(2) = m_{t_0,T}^{BS}(2) = [(r - q - \frac{\sigma^2}{2})(T - t_0)]^2 + \sigma^2(T - t_0) \tag{14}$$

Proof: Without any loss of generality, assume t_0=0 and q=0. Then from Eqs. (1)-(2), the first- and second-order RNMs are simplified as, respectively,

$$m_T(1) = e^{rT} - e^{rT}[\int_{S_0}^{\infty} \frac{1}{K^2} C(T,K)dK + \int_0^{S_0} \frac{1}{K^2} P(T,K)dK] - 1 \tag{1'}$$

$$m_T(2) = 2e^{rT}[\int_{S_0}^{\infty} \frac{1-\ln(K/S_0)}{K^2} C(T,K)dK + \int_0^{S_0} \frac{1-\ln(K/S_0)}{K^2} P(T,K)dK] \tag{2'}$$

<u>**A-1.**</u> *First,* we show that $m_T(1) = (r - \frac{\sigma^2}{2})T$.

Note that in B-S world, the call price is expressed by $C(T,K) = S_0 N(d_1) - Ke^{-rT} N(d_2)$, and put by $P(T,K) = -S_0 N(-d_1) + Ke^{-rT} N(-d_2)$, where $N(\bullet)$ is the cumulative distribution function of standard normal distribution. All other letters have the conventional meanings in the sense of Black-Scholes setting, hence not explained here, and purely for convenience, we denote $a=(r+\sigma^2/2)T$, $b=\sigma\sqrt{T}$ and $c=(r-\sigma^2/2)T$.

By direct calculations (double integrals involved),

$$\int_{S_0}^{\infty}\frac{1}{K^2}C(T,K)dK$$
$$=\frac{S_0}{K^2}\int_{S_0}^{\infty}N(d_1)dK-\frac{e^{-rT}}{K}\int_{S_0}^{\infty}N(d_2)dK$$
$$=[N(\frac{a}{b})-e^{-a+\frac{b^2}{2}}N(\frac{a}{b}-b)]-e^{-rT}[cN(\frac{c}{b})+\frac{b}{\sqrt{2\pi}}e^{-\frac{c^2}{2b^2}}]$$

Similarly,

$$\int_{0}^{S_0}\frac{1}{K^2}P(T,K)dK$$
$$=-e^{-rT}[cN(-\frac{c}{b})-\frac{b}{\sqrt{2\pi}}e^{-\frac{c^2}{2b^2}}]-[e^{-a+\frac{b^2}{2}}N(b-\frac{a}{b})-N(-\frac{a}{b})]$$

Summing them up and substituting the sum into (1’) yield,

$$m_T(1)=e^{rT}-e^{rT}[1-(e^{-a+\frac{b^2}{2}}+ce^{-rT})]-1=(r-\frac{\sigma^2}{2})T.$$

<u>A-2.</u> *Second*, the following is to prove $m_T(2)=[(r-\frac{\sigma^2}{2})T]^2+\sigma^2T$.

Although two more integrals, $\int_{S_0}^{\infty}\frac{ln(K)}{K^2}C(T,K)dK$ and $\int_{0}^{S_0}\frac{ln(K)}{K^2}P(T,K)dK$ are involved, it is not difficult but merely a little bit complicated. Using a similar way, it follows that

$$m_T(2)$$
$$=2e^{rT}[\int_{S_0}^{\infty}\frac{1-\ln(K/S_0)}{K^2}C(T,K)dK+\int_{0}^{S_0}\frac{1-\ln(K/S_0)}{K^2}P(T,K)dK]$$

$$= 2e^{rT}[\int_{S_0}^{\infty}\frac{1}{K^2}C(T,K)dK + \int_0^{S_0}\frac{1}{K^2}P(T,K)dK] - 2e^{rT}[\int_{S_0}^{\infty}\frac{\ln(K/S_0)}{K^2}C(T,K)dK + \int_0^{S_0}\frac{\ln(K/S_0)}{K^2}P(T,K)dK]$$

$$= (2e^{rT} - 2c - 2) + 2e^{rT}[\int_{S_0}^{\infty}\frac{\ln(S_0/K)}{K^2}(S_0N(d_1) - Ke^{-rT}N(d_2))dK + \int_0^{S_0}\frac{\ln(S_0/K)}{K^2}(-S_0N(-d_1) + Ke^{-rT}N(-d_2))dK]$$

$$= 2e^{rT} - 2c - 2 + 2e^{rT}[e^{-rT}\frac{b^2}{2} + e^{-rT}\frac{c^2}{2} - 1 + e^{-\frac{2a-b^2}{2}}(1+a-b^2)]$$

$$= [(r - \frac{1}{2}\sigma^2)T]^2 + \sigma^2 T$$

as desired.

B. *Now* return to the RNMs under B-S measure, $m_T^{BS}(1) = (r - \frac{\sigma^2}{2})T$ and $m_T^{BS}(2) = [(r - \frac{\sigma^2}{2})T]^2 + \sigma^2 T$.

Recall that, $m_T^{BS}(j) = E^{\pi^{BS}}([R_T]^j)$ *(j=1,2)* by definition, and $R_T = (\mu - \frac{\sigma^2}{2})T + \sigma\omega_T$ by formula (12).

Applying the Girsanov theorem [21], $\omega_t^{BS} := \omega_t - \frac{r-\mu}{\sigma}t$ ($0 \le t \le T$) is the standard Brownian motion under the B-S measure, then it is immediate that

$$m_T^{BS}(1) = E^{\pi^{BS}}[(\mu - \frac{\sigma^2}{2})T + \sigma\omega_T]$$

$$= (\mu - \frac{1}{2}\sigma^2)T + \sigma E^{\pi^{BS}}(\omega_T^{BS} + \frac{r-\mu}{\sigma}T),$$

$$= (r - \frac{1}{2}\sigma^2)T$$

and

$$m_T^{BS}(2) = E^{\pi^{BS}}([(\mu - \frac{\sigma^2}{2})T + \sigma\omega_T]^2)$$

$$= (\mu - \frac{1}{2}\sigma^2)^2T^2 + [2\sigma(\mu - \frac{1}{2}\sigma^2)T^2]E^{\pi^{BS}}(\omega_T^{BS} + \frac{r-\mu}{\sigma}T) + \sigma^2 E^{\pi^{BS}}[(\omega_T^{BS} + \frac{r-\mu}{\sigma}T)^2]$$

$$= [(r - \frac{1}{2}\sigma^2)T]^2 + \sigma^2 T$$

Thus, the proof is completed.

Apparently, from Theorem **2** the extracted RNMs from option prices through (1) and (2) are the same as the true RNMs under the B-S setting. Hence the RNMs are exactly risk-neutral with which the derived RND in (4) is consequently a risk-neutral martingale measure.

In addition, we further conduct simulations to check the accuracy of RNM estimates via formulae (1)~(2) following the procedures specified in Section 2.2.1, so that the correctness of obtained RNMs can be confirmed by comparing them with the corresponding true RNMs.

Given the initial time t_0=0, expiration T=1, interest rate r=0.05, dividend yield q=0.02, and volatilityσ=0.2, the B-S RNMs $m_{t_0,T}^{BS}(j)$ (j=1,2) can be easily calculated using (13)~(14). Meanwhile, according to Section 2.2.1, for an underlying asset's price S_0, numerically computing RNMs via integral expressions (1)~(2) requires several pairs of 'market-traded' options *C(T, K)* and *P(T, K)*. Within the B-S world, we then generate a set of OTM call and put options as the 'market-available' options. In the simulations, for fully checking the accuracy of estimating RNM, we consider five levels of underlying price $S_0 = 48, 50, 52, 54, 56$ and for each level, four pairs[8] of OTM options with 4-point increment strikes (See Table 1 as below) are generated as the 'real' market options, with which the RNMs $m_{t_0,T}(j)$ (j=1, 2) are calculated.

[8] We numerically calculate the RNMs according to integral expressions (1)~(2) by using various numbers of OTM options, the unreported results find no significant difference in the resultant estimates of RNM. Consider the accuracy of estimates and option-availability in real marketplace, four pairs of OTM options are chosen.

Table1 : Strike prices of generated 'market-observed' OTM options in a Black-Scholes world for a range of initial underlying prices.

Underling price S_0	48	50	52	54	56
Strikes of OTM calls	34, 38, 42, 46	36, 40, 44, 48	38, 42, 46, 50	40, 44, 48, 52	42, 46, 50, 54
Strikes of OTM puts	50, 54, 58, 62	52, 56, 60, 64	54, 60, 64, 68	56, 58, 62, 66	58, 62, 64, 70

Now with four pairs of OTM calls and puts for each underlying price above, for instance, (34, 50), (38, 54), (42, 58) and (46, 62) for 48, we can estimate RNMs following the procedures in Section 2.2.1 via trapezoidal rule integration method by setting K_∞=5$K_{\max}$, K_0=0.2K_{min} and the number of nonoverlapping subintervals m=80. Taking the integral $\int_{S_0}^{K_{\max}} \frac{1}{K^2} C(T,K)dK$ as an example, it can be accurately approximated as

$$\int_{S_0}^{K_{\max}} \frac{1}{K^2} C(T,K)dK \approx \frac{1}{2}[\sum_{i=1}^{m}(\frac{1}{K_{i-1}^{\ 2}} C(T,K_{i-1}) + \frac{1}{K_i^{\ 2}} C(T,K_i))\Delta K],$$

where $\Delta K = (K_{\max} - S_0)/m$, $K_i = S_0 + i\Delta K$ for $i \in [0,m]$ and $C(T,K_i)$ is obtained via interpolation using four available call prices. The estimates of RNM from (1)~(2) and the real values of RNM are shown in Table 2 for five underlying prices.

Table 2 : Comparisons between estimated RNM and real value in a Black-Scholes world for a range of initial underlying prices.

Underling price S_0	48	50	52	54	56
1st-order RNM	Real value: 0.0100				
	0.0100	0.0100	0.0100	0.0100	0.0100
2nd-order RNM	Real value: 0.0401				
	0.0401	0.0401	0.0401	0.0401	0.0401

Notes : The first two order moment estimates for log-return with various underlying's prices are compared to the theoretical values in the Black-Scholes market with parameters r=0.05, q=0.02,σ=0.2 and T=1. These moments are recovered by using only 4 pairs of options. For both moments, cells in the bottom row represent estimated values while the top is the real (theoretical) value.

As can be seen from Table 2, the RNM estimates are the same as the real (theoretical) values (retaining 4 digits after the decimal point). This demonstrates that four pairs of options can effectively capture the shape of the underlying distribution thanking to the accurate moment estimates. *Furthermore*, the estimated RNMs are almost indistinguishable for both moments, even though the underlying prices are different. This indicates that, as expected, two moments by (1)~(2) are exactly 'risk-neutral' and are not related to the growth rate μ and current asset price S_0. Conceptually, the risk-neutral moments of log-return in B-S world are only determined by the drift term and volatility term rather than the underlying asset price, which corresponds to formulae (13)~(14).

Risk-Neutrality of the Derived RND

So far the correctness of calculated RNMs $m_{t_0,T}(j)$ is already confirmed, then as an illumination of concept, the resultant probability measure $\hat{\pi}^Q = (\hat{\pi}_1^Q, \ldots, \hat{\pi}_N^Q)$—— as the risk-neutral pricing measure for option pricing, should be exactly 'risk-neutral'. To *further verify the risk-neutrality of measure* $\hat{\pi}^Q$, we depict two sets of risk-neutral probabilities $\hat{\pi}_i^{Q(1)}$ and $\hat{\pi}_i^{Q(2)}$ using two different growth rates and show the indistinguishableness by comparing the resultant risk-neutral probability distributions, as well as the estimated density functions. *First*, with the parameter setting ($t_0 = 0, T = 1, r = 0.05, q = 0.02, \sigma = 0.2$) in Section 3.1 and according to Eq.(12), two series of 365 $(T - t_0)$-period gross log-returns are produced using the risk-neutral growth rate ($\mu_1 = r = 5\%$) and an unrealistic rate ($\mu_2 = 100\%$), and then treated as the 'historical' log-returns R_i (*i=1,2,..., 356*) as defined in Section 2.1. *Second*, with these two series of historical log-returns, two corresponding RNDs for both cases are calculated as $\hat{\pi}_i^{Q(1)} = (\hat{\pi}_1^{Q(1)}, \hat{\pi}_2^{Q(1)}, \ldots, \hat{\pi}_{365}^{Q(1)})$ and $\hat{\pi}_i^{Q(2)} = (\hat{\pi}_1^{Q(2)}, \hat{\pi}_2^{Q(2)}, \ldots, \hat{\pi}_{365}^{Q(2)})$ via Eq.(4), respectively. *Finally*, we plot two sets of estimated risk-neutral probabilities $\hat{\pi}_i^{Q(1)}$ and $\hat{\pi}_i^{Q(2)}$ in Figure 1, and the corresponding risk-neutral cumulative distribution functions (CDFs) and probability density estimates (PDEs) are shown in Figure 2.

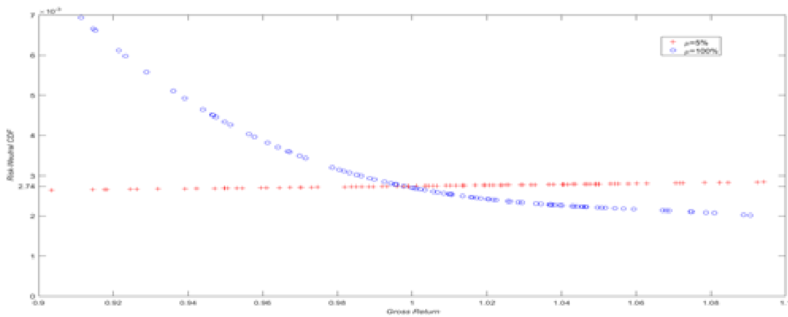

Figure: 1 Two sets of risk-neutral probabilities derived from GBMs with two drift rates of 5% and 100%. Note that, for clarity, only 70 points among the 365 historical gross returns are shown.

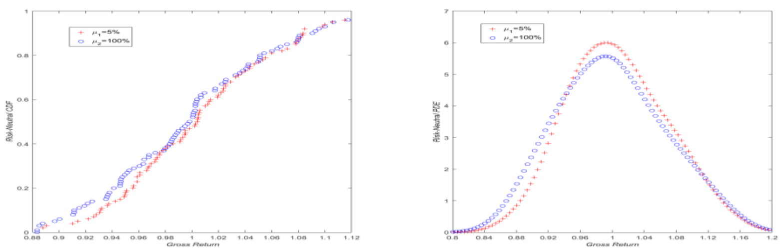

Figure: 2 Two risk-neutral cumulative distribution functions (CDFs) derived from GBMs with two growth rates of 6% and 100%, as well as the corresponding probability density estimates (PDEs). Note that only part of risk-neutral gross returns is illuminated for clarity.

Figure1 depicts the resultant risk-neutral probabilities $\hat{\pi}_i^{Q(1)}$ with a growth rate of 5% and $\hat{\pi}_i^{Q(2)}$ with 100% respectively, and *two major findings can be found* as expected. *First*, as can be seen from Figure1, all the risk-neutral probabilities are roughly equal to 0.00274 (=1/365) in the case of 5% growth rate. This is quite understandable since the input gross returns in this case are generated in the risk-neutral world where the growth rate is set to be the same as the interest rate at 5%. In contrast, the probability curve is decreasing with gross return when the growth rate is set at 100%. This is also explicable since the lower returns need to be assigned with higher probabilities so as to provide a risk-neutral measure. *Second*, although the curves of the probabilities for 5% and 100% growth rates look quite different, their corresponding cumulative distribution functions (CDFs) /probability density estimates (PDEs) are nearly indistinguishable (Figure 2). This is due to that two CDFs/PDFs must be very similar in order to yield approximately the same risk-neutral result. This result exactly implies the risk-neutrality of the probabilities.

Pricing Performance and Analysis

With the verifications of risk-neutrality RND, it is conceptually reasonable to claim that our proposed entropy-based scheme can provide an appropriate RND as the risk-neutral pricing measure and consequently a quite high pricing accuracy is ensured. This section processes a further evaluation of the proposed method by conducting simulation tests using two different drift rate μ in a Black-Scholes (B-S) environment, as well as in a more realistic stochastic volatility model of Heston [22]. Note that it is adequate to concentrate on the pricing of European call options for the following reasons: First, given the call price, the put price is immediate due to put-call parity. More importantly, formulae (9)-(10) tell that the same risk-neutral price paths are used when pricing call and put options which would also result in an accurate price for put option as the call is accurately priced.

Performance in a B-S Environment

In a B-S market, the GBM for underlying price S_t is givens by Eq. (11) and the log-return $R_{t_0,T}$ is calculated as (12). To make a *more comprehensive analysis*, various levels of moneyness (S_0/ K for call) and time to maturity are considered and we assign a reasonable set of parameter values as follows.

- Valuation date: t_0=0
- Expiration date (in year): T=1/12, 1/4, 1/2, 3/4, 1
- Strike price[9]: K=52
- Initial asset price: S_0 =48, 50, 52, 54, 56

[9] The unreported pricing results using other strike prices (e.g., K=50, 54 for different underlying price S_0) also show a high accuracy for option pricing, so only one case of K=52 is demonstrated in this study.

• Risk-free interest rate: r=5%
• Drift rate: μ_1=5%,μ_2=100%
• Volatility: σ=20%
• Dividend yield[10]: q=0

To enable the calculations of RNMs $m_{t_0,T}(1)$ and $m_{t_0,T}(2)$, as specified previously, four pairs of OTM options (with different strikes) are generated as the ‘real’ market options for each underlying price S_0. In this B-S setting, we use the strike prices in Table 2, e.g., four calls with strikes (34, 38, 42, 46) and four puts with (50, 54, 58, 62) to produce four pairs of call and put options as B-S market options to estimate the RNMs for each price of S_0 with a time to maturity T.

For each time to maturity T, 365 log-returns are drawn from (12), then following the operational steps detailed in Section 3.2, two corresponding RNDs with different drift rate are derived and used as the risk-neutral measures for pricing the options.

Table 3 reports the pricing results with two different growth rates using the proposed RNMs-based entropy method (Entropy). For each reported price, five independent simulations are carried out and the resultant prices averaged. In each simulation, 5,000 risk-neutral price paths are generated for computing the call price by using formula (9).

[10] Without any loss of generality but merely a computational convenience, here we set dividend yield q=0.

Table 3：Averaged price estimates of call options across a range of asset prices (with $K = 52$) in B-S world.

Asset price S_0	Option Price	Time to maturity (year)				
		1/12	1/4	1/2	3/4	1
48	B-S price (True value)	0.1271	0.7389	1.6321	2.4428	3.1934
	Our Entropy Estimate (μ_1=5%)	0.1272	0.7392	1.6319	2.4430	3.1920
	Difference (%)	0.0787	0.0406	-0.0123	0.0082	-0.0438
	Entropy (μ_2=100%)	0.1273	0.7395	1.6317	2.4437	3.1918
	Difference (%)	0.1574	0.0812	-0.0245	0.0368	-0.0501
50	B-S	0.4897	1.4155	2.4950	3.4107	4.2352
	Entropy (μ_1=5%)	0.4899	1.4161	2.4961	3.4095	4.2345
	Difference (%)	0.0408	0.0424	0.0441	-0.0352	-0.0165
	Entropy (μ_2=100%)	0.4902	1.4163	2.4964	3.4092	4.2351
	Difference (%)	0.1021	0.0565	0.0561	-0.0440	-0.0024
52	B-S	1.3063	2.3998	3.5821	4.5616	5.4343
	Entropy (μ_1=5%)	1.3067	2.4003	3.5816	4.5610	5.4331
	Difference (%)	0.0306	0.0208	-0.0140	-0.0132	-0.0221
	Entropy (μ_2=100%)	1.3071	2.4005	3.5810	4.5608	5.4312
	Difference (%)	0.0612	0.0292	-0.0307	-0.0175	-0.0570
54	B-S	2.6336	3.6830	4.8787	5.8812	6.7775
	Entropy (μ_1=5%)	2.6331	3.6821	4.8774	5.8785	6.7745
	Difference (%)	-0.0190	-0.0244	-0.0266	-0.0459	-0.0443
	Entropy (μ_2=100%)	2.6328	3.6825	4.8772	5.8784	6.7741
	Difference (%)	-0.0304	-0.0136	-0.0307	-0.0476	-0.0502
56	B-S	4.3418	5.2184	6.3578	7.3495	8.2483
	Entropy (μ_1=5%)	4.3407	5.2165	6.3563	7.3468	8.2449
	Difference (%)	-0.0253	-0.0364	-0.0236	-0.0367	-0.0412
	Entropy (μ_2=100%)	4.3410	5.2163	6.3564	7.3465	8.2445
	Difference (%)	-0.0184	-0.0402	-0.0220	-0.0408	-0.0461

Notes: The values in the first row are computed using Black-Scholes formula as the 'true' prices. 'Entropy' rows report the price estimates with the risk-neutral growth rates of 5% (i.e. $\mu_1 = r$) and 100% from our entropy method with the corresponding differences. 'Difference' rows measure the difference between price estimate of Entropy method and 'true' price (B-S price), which is calculated by dividing the estimated price minus the 'true' price by the 'true' price, that is, $(p^{estimate} - p^{true}) \times 100 / p^{true}$. For the reported price estimates from Entropy, each of them is the averaged value over five independent simulations and each of the simulations generates 5,000 sample price paths.

As is shown in Table 3, the resultant prices by our entropy method are amazingly close to the 'true' values (B-S prices) for both drift rates over a range of underlying prices. The differences in absolute value between the estimates and the B-S prices are all below 0.16% in the case of growth rate 100%, with the largest being 0.1574% for the option with asset price of 48 and a short term of 1/12. For the case of risk-neutral growth rate of 5%, all are below 0.08% with the largest only 0.0787%. It is quite understandable for the highest difference, considering that in this situation the option is out-of-the-money and the true value (B-S price) is very small at 0.1271 which could result in a 'big' error when computing the percentage difference.

Furthermore, for each price estimate in both growth rates of 5% and 100%, two pricing errors are so small that the difference between two estimates is slight. This finding again illustrates that this proposed method is independent of the drift rate and the resultant RND is actually risk-neutral, as previously outlied in Figure 1-2, which ensures high pricing precision. Consequently, it appears that the proposed RNMs-based entropy valuation is *completely comparable to the benchmarking Black-Scholes formula for European options.*

Performance in a Stochastic Volatility Model

In order to conduct a more realistic test of proposed entropy method, following the convention of many literatures (e.g., Haley & Walker, [23]), we investigate the performance of our method using Heston's stochastic volatility (SV) model (1993), where the asset price follows

$$\frac{dS_t}{S_t} = \mu dt + \sqrt{v_t}\, d\omega_{s,t} \tag{15}$$

and the variance of the return follows an Ornstein-Uhlenbeck process

$$dv_t = \kappa(\theta - v_t)dt + \eta\sqrt{v_t}\, d\omega_{v,t} \tag{16}$$

where as usual κ is the speed of mean reversion, θ the long-run variance, η the volatility of the volatility generating process, and $d\omega_{s,t}$ and $d\omega_{v,t}$ are Wiener processes with correlation ρ.

The appeal of this setup is that the model retains an integral-involved closed form solution for the European option price. To bypass the substantial bias [24], we adopt Gauss–Kronrod quadrature method [25] rather than the commonly-known Euler discretization when calculating the integrals, then the option price under this SV model is computed.

As did in Haley and Walker [23], the same parameter values for SV model (15)-(16) are given as below,

- Drift rate: μ=10%
- Mean reversion: κ=3
- Long-run mean: θ=4%
- Volatility: η=40%
- Correlation: ρ = -0.5

and the factors of option being valued are as follows,

- Valuation date: t_0=0
- Expiration date (in year): T=1/12, 1/4, 1/2, 3/4, 1
- Strike price: K=52
- Risk-free interest rate: r=5%
- Dividend yield: q=0

According to the procedures in Section 3.2 and following the same computational details as Section 4.1, a sample of log-return is generated using the parameter values above and the RNMs can be calculated so that the risk-neutral paths are simulated, ultimately the option price is computed. The pricing results are outlined in Table 4. Each price resulted from our entropy scheme (Entropy) is the averaged values over five independent simulations and each of simulations generates 5,000 risk-neutral sample price paths.

Table 4: Averaged price estimates of call options across a range of underlying prices (with $K = 52$) in Heston's SV model.

Underling price S_0	Option Price	Time to maturity				
		1/12	1/4	1/2	3/4	1
48	SV Mode (True value)	1.1963	2.7061	3.9327	4.7408	5.3936
	Our Entropy Scheme	1.1970	2.7075	3.9319	4.7427	5.3906
	Difference (%)	0.0611	0.0503	-0.0215	0.0405	-0.0552
50	SV	1.9506	3.6527	4.9873	5.8583	6.5574
	Entropy	1.9517	3.6553	4.9903	5.8558	6.5562
	Difference (%)	0.0547	0.0699	0.0609	-0.0434	-0.0185
52	SV	2.9381	4.7514	6.1631	7.0835	7.8203
	Entropy	2.9390	4.7527	6.1621	7.0846	7.8184
	Difference (%)	0.0302	0.0271	-0.0163	0.0155	-0.0249
54	SV	4.1476	5.9904	7.4492	8.4063	9.1727
	Entropy	4.1467	5.9888	7.4467	8.4021	9.1772
	Difference (%)	-0.0211	-0.0275	-0.0329	-0.0496	0.0492
56	SV	5.5532	7.3555	8.8344	9.8164	10.6052
	Entropy	5.5517	7.3524	8.8366	9.8123	10.6102
	Difference (%)	-0.0278	-0.0423	0.0249	-0.0416	0.0471

Notes: The values in the first row are obtained by assuming a Heston's SV model, and naturally regarded as the 'true' prices. The second reports the price estimates from our entropy scheme. The 'Difference' row measure the difference between price estimate of Entropy and 'true' price, which is calculated by dividing the estimated price minus the 'true' price by the 'true' price, that is, $(p^{estimate} - p^{true}) \times 100 / p^{true}$. For the reported price estimates from Entropy method, each of them is the averaged values over five independent simulations and each of the simulations generates 5,000 sample price paths.

Again, the comparisons facilitated in SV model exploit the superb pricing power of the proposed RNMs-constrained entropy scheme. First, observations from Table 4 show that the estimated price from entropy scheme is fairly close to the 'true' value (SV price) for each combination of asset price (or moneyness) and time to maturity, and this finding in SV model is in line with that in B-S world. Second, using the deviation judging measure---Difference indicator, the pricing error is definitely acceptable since the largest absolute difference value is merely 0.0611% in this more realistic circumstance. *More importantly*, the slight pricing error using diffenrence indicator reveals that the RNMs can also effectively capture the feature of risk-neutral distribution such as the volatility within Heston's SV model. It should be noted that, as does in B-S world, there is no discernible relation between the pricing accuracy and moneyness (S_0 / K) or time to maturity.

In brief, these pricing results described in Tables 3-4 indicate that European calls (thus the puts) can be priced rather accurately by our entropy approach in both simulated markets, regardless of an ideal environment or a more realistic model. It is noteworthy that the extracted informative RNMs from the option 'market' play a significant role for entropy approach to has the superb pricing performance, as RNMs can capture the shape of RND accurately enough.

Conclusion

This article establishes a risk-neutral moment-constrained (RNMs-constrained) entropic pricing framework, within which the optimal RND (an equivalent martingale measure) is achieved via the maximum entropy principle, as an appropriate risk-neutral pricing measure to produce a rather accurate prices for options.

The informative RNMs can be recovered from a set of market-available options and utilized to correctly capture the features of the RND such as the volatility, skewness, and kurtosis, etc. for option pricing. We provide the general expression for extracting the RNMs and prove that the calculated RNMs using the expressions are the same as the true values of RNM in a Black-Scholes setting. Further, the risk-neutrality of such obtained RND within the RNMs-based entropy frame is deeply verified in the simulation experiments by showing the independence of RND on the growth rate of underlying asset.

The pricing performance of our entropy pricing method is fully evaluated in simulation environments including a more realistic stochastic volatility (SV) scenario as well as the Black-Scholes market. The simulation testing, in a Black–Scholes (B-S) world, demonstrates that the resultant prices with both different drift rates from the entropy method are so close to the true values (B-S prices) that the pricing error for each option is too slight. Hence this entropy valuation sounds comparable to the right benchmark B-S formula in the B-S setting. Within the Heston's SV model, this entropy method again price options pretty well for a range of combinations of moneyness and time to maturity. The results under this SV model reveal that the pricing bias for each combination is so small that the difference between the price from our entropy scheme and that from SV model has no obvious discernible pattern with moneyness or maturity. It should also be noted that the imposed RNMs restrictions in our valuation are of great importance since the RNMs contain much useful market information so that volatility smile, skewness and excess kurtosis, can be effectively reflected in the derived risk-neutral martingale measure.

In summary, this proposed RNMs-constrained entropy valuation is conceptually and practically appealing since it does not impose any underlying structural assumption but relies more on the effective information contained in the marketplace, and in this way the resultant price of option can match the market behavior and be close enough to the actual market price of the option. Therefore, in principle, this entropy method can be applied in any other artificial circumstances and real markets due to its ability to achieve a martingale measure close enough to the correct risk-neutral measure. Therefore it is not unreasonable to imagine that this proposed RNMs-based entropy method provides an attractive and effective way for option pricing.

References

1. Harrison MJ, Kreps DM. Martingales and arbitrage in multiperiod securities markets. J. Econ. Theory. 1979; 20: 381–408.
2. Harrison MJ, Pliska SR. Martingales and stochastic integrals in the theory of continuous trading. Stoch. Process. Their Appl. 1981; 11: 215–260.
3. Aït-Sahalia Y, Lo A. Nonparametric estimation of state-price densities implicit in financial asset prices. J. Financ. 1998; 53: 499–547.
4. Garcia R, Gencay R. Pricing and hedging derivative securities with neural networks and a homogeneity hint. J. Econom. 2000; 94: 93–115.
5. Aït-Sahalia Y, Duarte J. Nonparametric option pricing under shape restrictions. J. Econom. 2003; 116: 9–47.
6. Stutzer M. A simple nonparametric approach to derivative security valuation. J. Financ. 1996; 51: 1633–1652.
7. Buchen PW, Kelly M. The maximum entropy distribution of an asset inferred from option prices. J. Financ. Quant. Anal. 1996; 31: 143–159.
8. Stutzer M. Simple entropic derivation of a generalized Black-Scholes model. Entropy. 2000; 2: 70–77.
9. Alcock J, Auerswald D. Empirical tests of canonical nonparametric American option-pricing methods. J. Futures Mark. 2010; 30: 509–532.
10. Neri C, Schneider L. A family of maximum entropy densities matching call option prices. Appl. Math. Financ. 2013; 20: 548–577.
11. Yu X, Liu Q. Canonical least-squares Monte Carlo valuation of American options: Convergence and empirical pricing analysis. Math. Probl. Eng. 2014; 2014.
12. Liu X, Zhou R, Xiong Y, Yang Y. Pricing interval European option with the principle of maximum entropy. Entropy. 2019; 21: 788.
13. Feunou B, Okou C. Good volatility, bad volatility and option pricing. J. Financ. Quant. Anal. 2019; 54: 695–727.
14. Day T, Lewis C. Stock market volatility and the information content of stock index options. J. Econom. 1992; 52: 267–287.

15. Jackwerth JC. Option-implied risk-neutral distributions and implied binomial trees: A literature review. J. Deriv. 1999; 7: 66–82.
16. Britten-Jones M, Neuberger A. Option prices, implied price processes, and stochastic volatility. J. Financ. 2000; 55: 839–866.
17. Jiang G, Tian Y. The model-free implied volatility and its information content. Rev. Financ. Stud. 2005; 18: 1305–1342.
18. Yu X, Yang L. Pricing American options using a nonparametric entropy approach. Discrete Dyn. Nat. Soc. 2014; 2014.
19. Ben-Tal A. The entropic penalty approach to stochastic programming. Math. Oper. Res. 1985; 10: 263–279.
20. Brandimarte P. Numerical Methods in Finance and Economics: A MATLAB Based Introduction, 2nd ed. New York: Wiley. 2006.
21. Shreve SE. Stochastic Calculus for Finance II-Continuous-Time Models. Berlin: Springer. 2004.
22. Heston SL. A closed-form solution for options with stochastic volatility with applications to bond and currency options. Rev. Financ. Stud. 1993; 6: 327–343.
23. Haley MR, Walker T. Alternative tilts for nonparametric option pricing. J. Futures Mark. 2010; 30: 983–1006.
24. Broadie M, Kaya Z. Exact simulation of stochastic volatility and other a_ne jump di_usion processes. Oper. Res. 2006; 54: 217–231.
25. Kahaner D, Moler C, Nash S. NumericalMethods and Software. Upper Saddle River: Prentice-Hall. 1989.
26. Gray P, Edwards S, Kalotay E. Canonical valuation and hedging of index options. Journal of Futures Markets. 2007; 27: 771–790.
27. Alcock J, Smith G. Non-Parametric American option valuation using Cressie-Read divergences, Australian Journal of Management. 2017; 42: 252–275.
28. Bakshi G, Kapadia N, Madan D. Stock return characteristics, skew laws, and the differential pricing of individual equity options. Review of Financial Studies. 2003; 16: 101-143.

29. Agmon N, Alhassid Y, Levine RD. An algorithm for finding the distribution of maximal entropy. Journal of Computational Physics. 1979; 30: 250–258.
30. Mead LR, Papanicolaou N. Maximum entropy in the problem of moments. Journal of Mathematical Physics. 1984; 25: 2404–2517.