Book Chapter

Resolvent, Natural, and Sumudu Transformations: Solution of Logarithmic Kernel Integral Equations with Natural Transform

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In this paper, the resolvent of an integral equation was found with natural transform which is a new transformation which converged to Laplace and Sumudu transformations, and the result was confirmed by the Sumudu transform. At the same time, a solution to the first type of logarithmic kernel Volterra integral equations has been produced by the natural transform.

1. Motivation

Solving partial or ordinary differential equations by the integral transformation method is the most skilled technique in the world of mathematics. For a function $f(t)$ defined in the $(-\infty, \infty)$ range, integral transformations generally are defined in the forms of

$$\begin{aligned}\oint [f(t)](s) &= \int_{-\infty}^{\infty} K(s, t) f(t) dt, \\ \oint [f(t)](u) &= \int_{-\infty}^{\infty} K(t) f(ut) dt, \\ \oint [f(t)](s, u) &= \int_{-\infty}^{\infty} K(s, t) f(ut) dt,\end{aligned}\quad (1)$$

where $K(s, t)$ is the nucleus of the transformation and s is a real or complex number independent of t . As the kernel function changes, the name of the integral transformation will also change. For example, if the kernel $K(s, t)$ is like the functions e^{-ist} , $t^{s-1}(st)$, $tJ_n(st)$, and e^{-st} , the name of the transformations will change such as Fourier, Mellin, Hankel, and Laplace, respectively. If equation (1) is equal to $K(t) = e^{-t}$, the integral transformation will be called the Sumudu (S) transform. For $f(t) > 0$ and any value of n , the generalized Laplace and Sumudu transformations are as follows:

$$\begin{aligned}L[f(t)] &= F(s) = s^n \int_0^{\infty} e^{-s^{n+1}t} f(s^n t) dt, \\ S[f(t)] &= G(u) = u^n \int_0^{\infty} e^{-u^n t} f(u^{n+1}t) dt.\end{aligned}\quad (2)$$

If $n = 0$ in these equations, Laplace and Sumudu transforms will be obtained. The $f(t) \in \mathbb{R}^2$ real function is a function defined in the set of

$$\begin{aligned}A &= \{f(t) \mid \exists M, \tau_1, \text{ and/or } \tau_2 > 0, \text{ such that } |f(t)| \\ &< M e^{|t|/\tau_j}, \text{ if } t \in (-1)^j \times [0, \infty), \quad j = 1, 2\},\end{aligned}\quad (3)$$

exponential order, sectionwise continuous with $f(t) > 0$ and $f(t) = 0$ for $t < 0$, where M is a finite constant number and τ_1 and τ_2 can be finite or infinite. The natural transformation (N -transform) of the function $f(t)$ for $t \geq 0$ is a function dependent on the s and u variables in the form of

$$N^+[f(t)] = R(s, u) = \int_0^{\infty} f(ut) e^{-st} dt, \quad \operatorname{Re}(s) > 0, \quad u \in (-\tau_1, \tau_2), \quad (4)$$

or

$$R(s, u) = \frac{1}{u} \int_0^{\infty} f(t) e^{-(st/ut)} dt, \quad (5)$$

where t is the time variable and s is the frequency variable. The discrete form of natural transformation is expressed as

$$N^+[f(t)] = R(s, u) = \sum_{n=0}^{\infty} \frac{n! a_n u^n}{s^{n+1}}. \quad (6)$$

Inverse natural transformation is defined as follows [1]:

$$N^{-1}[R(s, u)] = f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} R(s, u) e^{st/u} ds. \quad (7)$$

When $u \equiv 1$, equation (4) strictly converges to the Laplace transform:

$$L[f(t)] = F(s) = \int_0^{\infty} f(t) e^{-st} dt, \quad \text{Re}(s) > 0, \quad (8)$$

when $s \equiv 1$, equation (4) strictly converges to the Sumudu transform:

$$S[f(t)] = G(u) = \int_0^{\infty} f(ut) e^{-t} dt, \quad u \in (-\tau_1, \tau_2). \quad (9)$$

These can be expressed also by notation as follows:

$$\begin{aligned} N^+ f(u, s) &= N_f^+(u, s) = N_f^+ = N^+[f(t)], \\ N_f^+(u, 1) &= S_f^+(u), \\ N_f^+(1, s) &= L_f^+(s). \end{aligned} \quad (10)$$

Convolution theorem: the natural transformations of the functions $f(t)$ and $g(t)$ defined in the set A are $F(s, u)$ and $G(s, u)$, while the convolution of these functions is

$$N^+[(f * g)(t)] = uF(s, u)G(s, u). \quad (11)$$

Linearity property: if α, β any two constant and $f(x), g(x)$ any functions, natural transform provides the equality

$$N^+[\alpha f(t) + \beta g(t)] = \alpha F(s, u) + \beta G(s, u). \quad (12)$$

Integral equations occur in many areas of mechanics and mathematical physics. They also emerge as an equivalent representation of differential equations. That is, a differential equation can be replaced by an integral equation that matches the boundary conditions. Integral equations also constitute one of the most useful tools in many branches of pure analysis [2–4].

An integral equation is usually the following equations, where the unknown function appears under one or more integral symbols:

$$f(s) = \int_a^b K(s, t) g(t) dt, \quad (13)$$

$$g(s) = f(s) + \int_a^b K(s, t) g(t) dt. \quad (14)$$

In these equations $g(t)$ is an unknown function, others are known functions. These functions can be complex-valued functions depending on s and t real variables. When the unknown function is linear, equations (13) and (14) are called linear integral equations.

$$h(s)g(s) = f(s) + \lambda \int_a^b K(s, t) g(t) dt, \quad (15)$$

in which the upper limit can be a variable or constant is the most general form of the linear integral equation, where f, h , and K are known functions, g is the function to be found, and λ is a nonzero, real, or complex parameter. The function $K(s, t)$ is also called the kernel. Let us now examine the specific cases of equation (15):

(1) If the upper limit is a constant (b) as in equation (15), the equation is named Fredholm integral equation. This is divided into three:

(a) In case $h(s) = 0$ is called the first-kind Fredholm integral equation,

$$0 = f(s) + \lambda \int_a^b K(s, t) g(t) dt. \quad (16)$$

(b) In case $h(s) = 1$ is called the second-kind Fredholm integral equation:

$$g(s) = f(s) + \lambda \int_a^b K(s, t) g(t) dt. \quad (17)$$

(c) If $f(s) = 0$ is substituted in equation (17), it is called homogeneous Fredholm integral equation:

$$g(s) = \lambda \int_a^b K(s, t) g(t) dt. \quad (18)$$

(2) If in equations (16)–(18), the upper bound is a variable such as $b = s$, then these equations are called respectively the first-kind Volterra integral equation, the second-kind Volterra integral equation, and the homogeneous Volterra integral equation.

(3) If the kernel $K(s, t)$ is a univariate function of k that is a difference of $(s - t)$,

$$g(s) = f(s) + \lambda \int_a^b k(s - t) g(t) dt. \quad (19)$$

Integral equation is called the Fredholm integral equation of convolution type.

(4) The following equation

$$f(s) = \int_a^s \frac{g(t)}{(s - t)^a} dt, \quad s > a \quad (20)$$

is also called the Abel integral equation [5].

In equation (17), instead of kernel $K(s, t)$, if $R(s, t; \lambda)$ function that is so-called solvent core (resolvent) is represented, and the equation is converted to

$$g(s) = f(s) + \lambda \int_a^b R(s, t; \lambda) f(t) dt, \quad (21)$$

and so on in the right side of this equation, everything is clear, and the unknown function has disappeared. Therefore, by making integral process, $g(s)$ function which is the solution of the equation will be found.

2. Introduction

As with differential equations, there are many integral transformations such as Fourier, Laplace, Mellin, and Hankel to solve integral equations [2, 6, 7]. Fourier and Laplace transformations are the most common ones. Recently, a new integral transformation, called the natural transformation (N -transform), has been discovered that makes things easier. The characteristic of this new transformation is its convergence to Laplace and Sumudu transformations.

Now let us firstly talk about Sumudu transformation: the Sumudu transformation, which was considered to be new for this century, was discovered by Watugala in 1993 and the problems of control engineering were solved by this transformation [8]. The complex inverse formula was given by Weerakoon for this transformation [9]. Asiru applied the Sumudu transform to integral equations in his study [10] and systems of discrete dynamic equations in his study [11]. The study of Laplace-Sumudu Duality (LSD) is shown in [12]. The properties of the Sumudu transformation and a large list of the Sumudu transformation of functions are given by Belgacem and Karaballi [13]. Much more information about this transformation can be seen in [14].

Now, let us talk about the natural transformation: the introduction of natural transformation and the explanation of the use of the N -transform in a linear differential equation are studied by Khan and Khan [15]. The solution of Maxwell equations by natural transformation was obtained in the study [1]. Obtaining the N -transformation from the Fourier integral, becoming theoretical duals of the Laplace and Sumudu transformations, natural-multiple shift theorems, Bromwich contour integral, and Heaviside's expansion formula for inverse N -transformation are described in detail [16]. Complex inverse natural transformation and Heaviside's expansion formula together with relation to the N -transformation of Bessel function is described in [17]. The initial value problems with constant coefficients were solved by this transform in [18], and the situation of N -transformation in generalized functions space was discussed. Loonker and Banerji solved the first- and second-kind of convolution type Volterra integral equations and the Abel integral equation by using natural transformation and defined the solutions obtained in this way in certain distribution spaces [5]. The Laplace, Sumudu, Fourier, and Mellin transformations were obtained from natural transformation by Shah et al. [19]. Many more studies have been carried out and continue to be done quickly. One of the recent studies is the solution of the airy differential equation with natural transformation [20].

In this study, the solvent nucleus (resolvent) of an integral equation was found by natural transformation and the formula was confirmed by the Sumudu transform. At the same time, a solution of the first type of logarithmic core Volterra integral equations has been produced by natural transformation.

3. Calculating the Solver Kernel of the Second-Type Volterra Integral Equation with Convolution Type Kernel by Natural Transform

Such equations are

$$g(x) = f(x) + \int_0^x k(s-t)g(t)dt. \quad (22)$$

Let $N[g(x)] = G(s, u)$, $N[f(x)] = F(s, u)$, $N[k(s-t)] = K(s, u)$, and $\lambda = 1$. If the N -transform, convolution theorem, and linearity of the transform are applied to both sides of equation (22),

$$N[g(x)] = N[f(x)] + N\left[\int_0^x k(s-t)g(t)dt\right],$$

$$G(s, u) = F(s, u) + uK(s, u)G(s, u), \quad (23)$$

$$G(s, u) = \frac{F(s, u)}{1 - uK(s, u)}, \quad uK(s, u) \neq 1,$$

is obtained.

If the resolvent of the second-type Volterra integral equation is $R(x, t)$, then the equation is as follows:

$$g(x) = f(x) + \int_0^x R(x-t)f(t)dt. \quad (24)$$

Let $N[R(x-t)] = r(s, u)$, when N -transformation on both sides of equation (24) and then convolution theorem is applied:

$$N[g(x)] = N[f(x)] + N\left[\int_0^x R(x-t)f(t)dt\right],$$

$$G(s, u) = F(s, u) + u.r(s, u)F(s, u), \quad (25)$$

$$r(s, u) = \frac{G(s, u) - F(s, u)}{u.F(s, u)},$$

are obtained. If the value of $G(s, u)$ in equation (23) is written in equation (25),

$$r(s, u) = \frac{K(s, u)}{1 - u \cdot K(s, u)}, \quad (26)$$

and if applied to both sides N^{-1} of this equation, the resolvent of equation (22) is calculated in the following form:

$$N^{-1}[r(s, u)] = R(x, t) = N^{-1}\left[\frac{K(s, u)}{1 - u.K(s, u)}\right]. \quad (27)$$

Ex1: let us calculate the resolvent of the second-type Volterra integral equation by N -transformation such as the

kernel function calculated by $\sin(x-t)$, the parameter by 1, and the homogeneous disrupting function by x ;

The aforementioned integral equation is

$$g(x) = x + \int_0^x \sin(x-t)g(t)dt, \quad (28)$$

$$N[k(x-t)] = N[\sin(x-t)] = K(s, u) = \frac{u}{s^2 + u^2},$$

which is known from the N -transform tables. According to equation (27),

$$R(x, t) = N^{-1} \left[\frac{(u/s^2 + u^2)}{1 - u \cdot (u/s^2 + u^2)} \right] = N^{-1} \left[\frac{u}{s^2} \right] = t. \quad (29)$$

It is easily seen that the founded $R(x, t)$ provides the solution

$$g(x) = x + \int_0^x t \cdot t \cdot dt = x + \frac{x^3}{3}. \quad (30)$$

Ex2: let us calculate notationally the resolvent of the second-type Volterra integral equation with logarithmic kernel by N -transformation.

The integral equation is

$$g(x) = f(x) + \int_0^x \ln(t-x)g(t)dt. \quad (31)$$

According to $N[\ln(t-x)] = -(1/s)\log(u/s) + (\gamma/s)$ and (27),

$$R(x, t) = N^{-1} \left[\frac{-(1/s)\log(u/s) + (\gamma/s)}{1 - (-(1/s)\log(u/s) + (\gamma/s))} \right], \quad (32)$$

where γ is the Euler–Mascheroni constant.

4. Determination of Resolvent of the Type 2 Volterra Integral Equation with Convolution Type Kernel by Sumudu Transformation

The equations mentioned in the title are the following equations:

$$g(x) = f(x) + \int_0^x k(x-t)g(t)dt. \quad (33)$$

Let $S[g(x)] = G(u)$, $S[f(x)] = F(u)$, and $S[k(x-t)] = K(u)$. If the Sumudu transform and then convolution theorem are applied to both sides of equation (33),

$$S[g(x)] = S[f(x)] + S \left[\int_0^x k(x-t)g(t)dt \right],$$

$$G(u) = F(u) + uK(u)G(u), \quad (34)$$

$$G(u) = \frac{F(u)}{1 - u \cdot K(u)}, \quad u \cdot K(u) \neq 1,$$

where $F(u)$ and $K(u)$ are, respectively, the cases taken as $s = 1$ in the N -transformation of $f(x)$ and $k(x-t)$ functions. The solution is $g(t) = N^{-1}[F(u)/(1 - u \cdot K(u))]$.

If the resolvent of the second-type Volterra integral equation is represented by $R(x, t)$, the equality

$$g(x) = f(x) + \int_0^x R(x-t)f(t)dt \quad (35)$$

is provided. If the S transformation and convolution theorem are applied on both sides of equation (35), the following equations

$$S[g(x)] = S[f(x)] + S \left[\int_0^x R(x-t)f(t)dt \right],$$

$$G(u) = F(u) + u \cdot r(u)F(u), \quad (36)$$

$$r(u) = \frac{G(u) - F(u)}{u \cdot F(u)},$$

are obtained. If the value of $G(u)$ in equation (34) put in place in formula (36),

$$r(u) = \frac{K(u)}{1 - u \cdot K(u)}. \quad (37)$$

If S^{-1} is applied to both sides of this equation, the resolvent is

$$S^{-1}[r(u)] = R(x, t; 1) = R^{-1} \left[\frac{K(u)}{1 - u \cdot K(u)} \right]. \quad (38)$$

It is seen that formula (38) is the same as formula (27).

5. Obtaining the Solution of the First-Type Volterra Integral Equation with Logarithmic Kernel by Natural Transformation

Let the equation be

$$\int_0^x \ln(x-t)g(t)dt = f(x), \quad f(0) = 0. \quad (39)$$

Assuming $N[g(t)] = G(s, u)$, $N[f(x)] = F(s, u)$, and $N[\ln(x-t)] = K(s, u)$, let us apply the N -transform on both sides of this integral equation:

$$N \left[\int_0^x \ln(x-t)g(t)dt \right] = N[f(x)]. \quad (40)$$

According to the convolution theorem,

$$u \cdot N[\ln(x-t)] \cdot N[g(t)] = N[f(x)],$$

$$u \cdot K(s, u) \cdot G(s, u) = F(s, u), \quad (41)$$

are written. Let us first compute the term $K(s, u)$.

We know that $N[t^{n-1}/\Gamma(n)] = (u^{n-1}/s^n)$, $n > 0$, from the natural transformation tables. We can also write this equation as $N[t^{n-2}] = (\Gamma(n-1) \cdot u^{n-2}/s^{n-1})$, $n > 0$. Let us derive this relation according to variable n :

$$N[t^{n-2} \cdot \ln t] = \frac{u^{n-2}}{s^{n-1}} \cdot \Gamma(n-1) \left(\frac{(d\Gamma(n-1)/dn)}{\Gamma(n-1)} + \ln u + \ln \frac{1}{s} \right). \quad (42)$$

For $n = 2$,

$$N[\ln t] = \frac{1}{s} \left(\frac{\Gamma'(1)}{\Gamma(1)} + \ln \frac{u}{s} \right), \quad (43)$$

where $(\Gamma'(1)/\Gamma(1)) = -\gamma$ is the Euler–Mascheroni constant. That is,

$$N[\ln t] = \frac{1}{s} \left(-\gamma + \ln \frac{u}{s} \right) = -\frac{(\gamma - \ln(us^{-1}))}{s} = K(s, u). \quad (44)$$

If we write this statement in equation (41), equality

$$G(s, u) = \frac{su^{-1} \cdot F(s, u)}{-\gamma + \ln(us^{-1})}, \quad (45)$$

is obtained. Multiply the second side of this equation by the term su^{-1}

$$G(s, u) = \frac{s^2}{u^2} \cdot \frac{F(s, u)}{su^{-1}[-\gamma + \ln(us^{-1})]}. \quad (46)$$

Add and subtract the term,

$$\frac{f'(0)}{u \cdot s \cdot u^{-1}[-\gamma + \ln(us^{-1})]}, \quad (47)$$

to this expression, and the equality

$$\begin{aligned} G(s, u) &= \frac{s^2}{u^2} F(s, u) \frac{1}{su^{-1}[-\gamma + \ln(us^{-1})]} - \frac{f'(0)}{u} \\ &\quad \cdot \frac{1}{s \cdot u^{-1}[-\gamma + \ln(us^{-1})]}, \\ &\quad + \frac{f'(0)}{u} \cdot \frac{1}{s \cdot u^{-1}[-\gamma + \ln(us^{-1})]}, \\ G(s, u) &= \left[\frac{s^2}{u^2} F(s, u) - \frac{f'(0)}{u} \right] \left[\frac{1}{su^{-1}[-\gamma + \ln(us^{-1})]} \right] \\ &\quad + f'(0) \cdot \left[\frac{1}{s[-\gamma + \ln(us^{-1})]} \right], \end{aligned} \quad (48)$$

is obtained. Let us apply the operator N^{-1} on both sides of this equation

$$\begin{aligned} N^{-1}\{G(s, u)\} &= N^{-1} \left\{ \left[\frac{s^2}{u^2} F(s, u) - \frac{f'(0)}{u} \right] \left[\frac{1}{su^{-1}[-\gamma + \ln(us^{-1})]} \right] \right\} \\ &\quad + N^{-1} \left\{ f'(0) \left[\frac{1}{s[-\gamma + \ln(us^{-1})]} \right] \right\}, \\ g(t) &= N^{-1} \left\{ \frac{s^2}{u^2} F(s, u) - \frac{f'(0)}{u} \right\} \cdot N^{-1} \left\{ \frac{1}{su^{-1}[-\gamma + \ln(us^{-1})]} \right\} \\ &\quad + N^{-1}\{f'(0)\} \cdot N^{-1} \left\{ \frac{1}{s[-\gamma + \ln(us^{-1})]} \right\}. \end{aligned} \quad (49)$$

We know that the natural transformation of the derivative is

$$N^+[f^{(n)}(t)] = R_n(s, u) = \frac{s^n}{u^n} R(s, u) - \sum_{k=0}^{n-1} \frac{s^{n-(k+1)}}{u^{n-k}} f^{(k)}(0), \quad (50)$$

[15]. In the problem, we can write

$$N[f''(t)] = \frac{s^2}{u^2} F(s, u) - \frac{f'(0)}{u}, \quad (51)$$

because $f(0) = 0$. Therefore, equation (49) can be written again as

$$\begin{aligned} g(t) &= N^{-1}\{N[f''(t)]\} \cdot N^{-1} \left\{ \frac{u}{s[-\gamma + \ln(us^{-1})]} \right\} \\ &\quad + f'(0) \cdot N^{-1} \left\{ \frac{1}{s[-\gamma + \ln(us^{-1})]} \right\}. \end{aligned} \quad (52)$$

Now, integrate both sides of the equality $N[t^{n-1}/\Gamma(n)] = (u^{n-1}/s^n)$, $n > 0$ from 1 to ∞ :

$$\begin{aligned} \int_1^\infty N \left[\frac{t^{n-1}}{\Gamma(n)} \right] dn &= \int_1^\infty \frac{u^{n-1}}{s^n} dn = \frac{1}{u} \int_1^\infty \left(\frac{u}{s} \right)^n dn \\ &= \frac{-1}{s[\ln u - \ln s]}. \end{aligned} \quad (53)$$

Now, let us calculate the integral $\int_1^\infty N[t^{n-1} a^{-n+1}/\Gamma(n)] dn$; for this, we first consider the integrant

$$N \left[\frac{t^{n-1} a^{-n+1}}{\Gamma(n)} \right] = \frac{a^{-n+1}}{\Gamma(n)} N[t^{n-1}] = \frac{a^{-n+1}}{\Gamma(n)} \Gamma(n) \frac{u^{n-1}}{s^n} = \frac{a}{u} \left(\frac{u}{as} \right)^n. \quad (54)$$

If we write the result instead of the integrant and take the integral,

$$\int_1^{\infty} N \left[\frac{t^{n-1} a^{-n+1}}{\Gamma(n)} \right] dn = \frac{a}{u} \int_1^{\infty} \left(\frac{u}{as} \right)^n dn = \frac{1}{s \cdot [\ln a + \ln s - \ln u]}, \quad (55)$$

is obtained. Here, we take $a = e^{\gamma}$

$$\int_1^{\infty} N \left[\frac{t^{n-1} e^{\gamma(-n+1)}}{\Gamma(n)} \right] dn = \frac{1}{s[\gamma + \ln(su^{-1})]}. \quad (56)$$

Let us apply N^{-1} to this equation:

$$\int_1^{\infty} \frac{t^{n-1} e^{\gamma(-n+1)}}{\Gamma(n)} dn = N^{-1} \left\{ \frac{-1}{s \cdot [-\gamma + \ln(u \cdot s^{-1})]} \right\}. \quad (57)$$

Let us write equality (57) in equality (53):

$$g(t) = -u \cdot f''(t) \int_1^{\infty} \frac{t^{n-1} e^{\gamma(-n+1)}}{\Gamma(n)} dn - f'(0) \int_1^{\infty} \frac{t^{n-1} e^{\gamma(-n+1)}}{\Gamma(n)} dn. \quad (58)$$

If the convolution theorem is applied to the first term to the right of this equation,

$$g(t) = - \int_1^{\infty} f''(t) \left(\int_1^{\infty} \frac{t^{n-1} e^{\gamma(-n+1)}}{\Gamma(n)} dn \right) dt - f'(0) \int_1^{\infty} \frac{t^{n-1} e^{\gamma(1-n)}}{\Gamma(n)} dn, \quad (59)$$

can be written.

Specifically, if $f(x) = x$ is taken,

$$g(t) = - \int_1^{\infty} \frac{t^{n-1} e^{\gamma(1-n)}}{\Gamma(n)} dn, \quad (60)$$

is obtained.

Ex3: let us calculate the solution of the second-type Volterra integral equation with the logarithmic kernel by the N -transform formally.

The integral equation is

$$g(x) = f(x) + \int_x^x \ln(t-x)g(t)dt. \quad (61)$$

$N[\ln(t-x)] = -(1/s)\log(u/s) + (\gamma/s)$ equality is known from the references:

$$N[g(x)] = N[f(x)] + N \left[\int_0^x \ln(t-x)g(t)dt \right],$$

$$G(s, u) = F(s, u) + u \left[-\frac{1}{s} \log\left(\frac{u}{s}\right) + \frac{\gamma}{s} \right] G(s, u), \quad (62)$$

$$G(s, u) = \frac{F(s, u)}{1 + (u/s)[\log(u/s) - \gamma]}.$$

If N^{-1} transformation is applied to both sides of the equation,

$$g(t) = N^{-1} \left[\frac{F(s, u)}{1 + (u/s)[\log(u/s) - \gamma]} \right], \quad (63)$$

is obtained.

Data Availability

No dataset and material were used in this study.

Conflicts of Interest

The author declares no conflicts of interest.

Authors' Contributions

All interpretations and explanations belong to the author. The author read and approved the final manuscript.

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