Book Chapter

A Solution of Airy Differential Equation via Natural Transform

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Abstract

Integral transformations and special functions have an important place in engineering mathematics. In addition to well-known transformations such as Fourier, Laplace and Mellin, a new

transformation called Natural Transformation which is more general form of Laplace transformation, has brought a new breath to solve the problems that arise in engineering applications. In this work the Airy differential equation which is important in physical sciences is solved by Natural Transform and Airy function are obtained.

Keywords

Airy Function; Natural Transform; Electromagnetic Theory

Introduction

Airy differential equation named after British mathematician and astronomer George Biddell Airy (1801-1892) is a special differential equation has been widely acknowledged by scientist all over the world since it constitutes a classical equation of mathematical physics. It has a wide range of applications, including but not restricted to modeling the diffraction of light near the caustic surface such as rainbow and optics problem.

The Airy differential equation, in fact, is a special case of Schrodinger's equation for a particle confined within a triangular potential well and for a particle in a one-dimensional constant force field. The Airy function which is the solutions of the Airy differential equation is also important in microscopy and astronomy; it describes the pattern due to diffraction and interference, produced by a point source of light. The solutions to a large number of problems may be expressed in terms of Airy function. One such problem is linearized Korteweg-de Vries equation [1,2].

Integral transforms such as Laplace, Fourier etc., are efficient alternative methods to solve ordinary, partial differential, and integral equations. In this context, the Natural transform which is used since the early 2000s brings a new breath to the solution of differential equations [3]. It has a strong theoretical background and also is relevant to other transforms in the literature. The most detailed form of this transformation has been examined by Belgacem and Silambarasan [4] and in this work, the relations of

NT to Fourier, Laplace, and Sumudu transformations has been discussed. The natural transforms also contributed to the solution of Volterra and Abel integral equations [5,6]. This transformation has been applied to the zero-order first kind Bessel function in [7] and to the q-Bessel functions in [8]. Al-Omari defined the Parseval-type equation of the NT in [9] and applied this transform to solving initial value problems [10]. In addition, solutions of fractional ordinary differential equations were discussed in [11-14]. Baskonus et al. [15] proposed a new method by combining the NT and Decomposition Method for some partial differential equations. For fractional ordinary differential equations, Rida et al.

[12] obtained a new technique by combining a domain decomposition method and natural transform method and Khan and Shah [13] gave analytic solution via NT. With the combination of Homotopy Perturbation Method and NT is reconsidered for fractional partial differential equation [14,16-18]. In [19], three problems of Newtonian Fluid flow on an infinite plate were solved by using NT.

In this work, the Natural transform (NT) is applied to the Airy differential equation for expressing a physical phenomenon in terms of an effective analytical form.

Natural Transform

NT, previously called N-transform [3], has been extensively researched by authors of [4]. Since NT combines the features of Laplace and Sumudu transforms, the region of converges includes both of them. Hence, the NT of the function is given by the following integral equation,

$$N^+[f(t)] = R(s, u) = \int_0^\infty e^{-st} f(ut) dt, \quad s > 0, u > 0 \quad (1)$$

where the variables s, u are the NT variables [2,3]. Let the function $f(t)$ is defined in the following set,

$$A = \{f(t) | \exists M, \tau_1, \tau_2 > 0, |f(t)| < Me^{\frac{|t|}{\tau_j}}, t \in (-1)^j \times [0, \infty), j = 1, 2\} \quad (2)$$

For $u \equiv 1$, the Eq.(1) strictly converges to the Laplace transform $F(s)$;

$$L[f(t)] = F(s) = \int_0^\infty e^{-st} f(t) dt, \quad (3)$$

and for $s \equiv 1$, the Eq.(1) strictly converges to the Sumudu transform $G(u)$;

$$S^+[f(t)] = G(u) = \int_0^\infty e^{-t} f(ut) dt. \quad (4)$$

While giving NT related features, we assume that the functions are defined in positive axis $(0, \infty)$ and it is worth mention that, t and u are time variable, s is frequency variable. Having background to NT certain analytical properties [2,3,4,6] are shown without proof in below there.

Properties of NT

Theorem 1: (Convolution) Let $F(s, u)$ and $G(s, u)$ are the NT of $f(t)$ and $g(t)$, respectively. Both defined in set A , then

$$N^+[(f * g)] = uF(s, u)G(s, u), \quad (5)$$

where $f * g$ is convolution of two functions defined by

$$\int_0^t f(a)g(t-a)da = \int_0^t f(t-a)g(a)da. \quad (6)$$

Theorem 2: (Inverse NT) $R(s, u)$ is the NT of function $f(t)$ in A , then its inverse NT is defined by

$$N^{-1}[R(s, u)] = f(t) = \lim_{T \rightarrow \infty} \frac{1}{2\pi i} \int_{\gamma - iT}^{\gamma + iT} e^{\frac{st}{u}} R(s, u) ds \quad (7)$$

where the integral is taken along $s = \gamma$ in complex plane $s = x + iy$. The real number γ is chosen so that $s = \gamma$ lies on right of all (finite (or) countably infinite) singularities.

Application of Natural Transform to Airy Ordinary

Differential Equation

G.B. Airy was particularly involved in optics, for this reason he was also interested in the calculation of light intensity in the neighborhood of a caustic. For this purpose, he introduced the function defined by the integral

$$W(m) = \int_0^{\infty} \cos \left[\frac{\pi}{2} (w^3 - mw) \right] dw \quad (8)$$

which is the solution of the following differential equation

$$W'' + \frac{\pi^2}{12} mW = 0 \quad (9)$$

In 1928, Jeffreys introduced the notation used

$$Ai(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i \left(x + \frac{t^2}{3} \right)} dt = \frac{1}{\pi} \int_0^{\infty} \cos \left(xt + \frac{t^3}{3} \right) dt \quad (10)$$

which is the solution of the following homogeneous ODE called

Airy DE

$$y'' - xy = 0 \quad (11)$$

The following Bairy function is another solution for Airy DE which differs from Airy function in phase by $\frac{\pi}{2}$

$$Bi(x) = \frac{1}{\pi} \int_0^{\infty} \left[e^{\left(xt - \frac{t^3}{3} \right)} + \sin \left(xt + \frac{t^3}{3} \right) \right] dt. \quad (12)$$

Applications of the NT to both sides of equation (11) gives

$$N[y''] - N[xy] = N[0]. \quad (13)$$

Then

$$\frac{s^2}{u^2} R(s, u) - \frac{s}{u^2} y(0) - \frac{1}{u} y'(0) - \frac{u}{s} \frac{d}{du} (uR(s, u)) = 0, \quad (14)$$

$$\frac{dR(s, u)}{du} + \left(\frac{u^3 - s^3}{u^4} \right) R(s, u) = -\frac{s^2}{u^4} y(0) - \frac{s}{u^3} y'(0). \quad (15)$$

If $y(0) = c_1$ and $y'(0) = c_2$ are taken in nonlinear equation (15), the following equation is obtained;

$$\frac{dR(s, u)}{du} + \left(\frac{u^3 - s^3}{u^4} \right) R(s, u) = -\frac{c_1 s^2}{u^4} - \frac{c_2 s}{u^3}. \quad (16)$$

Solving this, yields

$$R(s, u) = \frac{1}{u} e^{-\frac{s^3}{3u^3}} \left(c(s) + \frac{c_1 s^2 \left(\Gamma\left(\frac{2}{3}\right) - \gamma\left(\frac{2}{3}, -\frac{s^3}{3u^3}\right) \right)}{3u^2 \left(-\frac{s^3}{3u^3} \right)^{2/3}} - \frac{c_2 s \left(\gamma\left(\frac{1}{3}, -\frac{s^3}{3u^3}\right) - \frac{2\pi\sqrt{3}}{3\Gamma\left(\frac{2}{3}\right)} \right)}{3u \left(-\frac{s^3}{3u^3} \right)^{1/3}} \right). \quad (17)$$

where γ is incomplete gamma function [20]. When we take the integral constant $c(s)$ as zero and make the necessary adjustments, we find,

$$R(s, u) = \frac{1}{3^{2/3}u} e^{-\frac{s^3}{3u^3}} \left(3^{1/3} c_1 \left(\Gamma\left(\frac{2}{3}\right) - \gamma\left(\frac{2}{3}, -\frac{s^3}{3u^3}\right) \right) + c_2 \left(\gamma\left(\frac{1}{3}, -\frac{s^3}{3u^3}\right) - \frac{2\pi\sqrt{3}}{3\Gamma\left(\frac{2}{3}\right)} \right) \right). \quad (18)$$

By applying the invers NT to $R(s, u)$ gives

$$N^{-1}[R(s, u)] = y(x) \quad (19)$$

$$y(x) = c_1 \left(1 + \frac{x^3}{3.2} + \frac{x^6}{6.5.3.2} + \frac{x^9}{9.8.6.5.3.2} + \dots \right) + c_2 \left(x + \frac{x^4}{4.3} + \frac{x^7}{7.6.4.3} + \frac{x^{10}}{10.9.7.6.4.3} + \dots \right) \quad (20)$$

$$y(x) = c_1 Ai(x) + c_2 Bi(x) \quad (21)$$

where Ai and Bi are Airy functions, at the same time, they are equivalent to the Maclaurin series of the Airy function.

Conclusion

In this work Natural Transform based on Laplace Transform is implemented to solve the Airy differential equation which is important in mathematical physics. It can be concluded that NT is a very efficient and alternative method in finding an exact solution for the Airy differential equation, other differential equations, and integral equations.

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