Book Chapter

Theories of Probability, Information and Graphs in Applied Geophysics

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Abstract

The possible ways for formalization of geophysical-geological investigations are outlined. It is shown that all the available geophysical-geological information can be represented by the classic three-level model. The main aim of the paper is a problem of determination of set of means composing the notion "geophysical-geological prospecting" (relative to some fixed feature) by assumed reliabilities of the means. The reliability of geological prospecting means is considered at the level of local determination. Reliabilities of information obtaining by separate mean and set of means are analyzed in detail. Suggested procedure of determining reliability for means and sets of means relative to feature are based on improved methodology of conditional probability utilization. The ways providing the increment of reliability of geological means are proposed. The applicability of proposed methods is shown on simplified examples. Estimating the efficiency of individual geophysical methods and their combination is analyzed. Practical employment of probabilities of type I and type II errors are shown. A selection of number of geophysical methods to solving different problems usually has no theoretical substantiation. The solution to this "four color" mathematical problem enables to assume that two independent geophysical methods are sufficient theoretically to characterize the geological-geophysical peculiarities of the area under study.

Keywords

Probabilistic Estimations, Increment of Reliability Informational Optimization, Entropy, Informativeness, Geophysical Method Integration, Geophysical Map Coloring

Introduction

It is well-known that majority of the inverse problem solutions in geophysics are ill-posed (e.g., [1,2]). It means, according to Hadamard [3], that the solution does not exist, or is not unique, or is not a continuous function of observed geophysical data (when small perturbation in the observations will cause arbitrary mistake in the solution). This fact, in particular, calls to wide application of informational and probabilistic methodologies in applied geophysics. Main results of geophysical data measurements, processing and interpretation are usually reflected in various maps of different kinds, patterns and scales. Thus, the informational and probabilistic methods application in applied geophysics could have a greatest importance. Theory of graphs occupies in geophysics a separate place.

Formalization of Geological-Geophysical Investigations

Geophysical maps are one of the most important tools for the condensed and effective geophysical-geological (seismological, ecological, atmospheric, etc.) data presentation (Figure 1). The compiled maps are employed at different stages (levels) of common and concrete investigations. It is necessary to underline that the same data may be used for different aims; at the same time different data may be applied for solving the same problem.

Geophysical observations are notoriously complicated by numerous factors (complex surrounding media, uneven topography, oblique polarization, instrumental errors, etc.) [4,5]. To eliminate many of these disturbances modern interpretational methodologies have been developed (e.g., [6-8]). However, at times the complexity of the geological environment (extreme variability in lateral and vertical physical properties), the presence of several anomalous targets (*AT*) in close proximity and additional disturbances makes it impossible or unfeasible to apply these methodologies. In such cases information-probabilistic methods are effective tools to recognize and classify targets, estimate the potential information value of geophysical methods and decide upon a workable solution. The objective of geophysical surveys application is to obtain qualitative and quantitative information about the geometric and physical characteristics of buried objects; e.g., to develop physical-geological models (*PGM*) of target objects. *PGM*s of varying degrees of complexity (the simplest *PGM*s are simply target identification and complex *PGM*s can be 3D models of the objects under study) can be used for substantiation of different types of industrial (drilling, excavation, exploitation, building, etc.) and scientific (geophysical monitoring, geological-geophysical mapping, construction of new *PGM*s, etc.) activity, and generation of future strategies of geological-geophysical investigations in the areas under study.

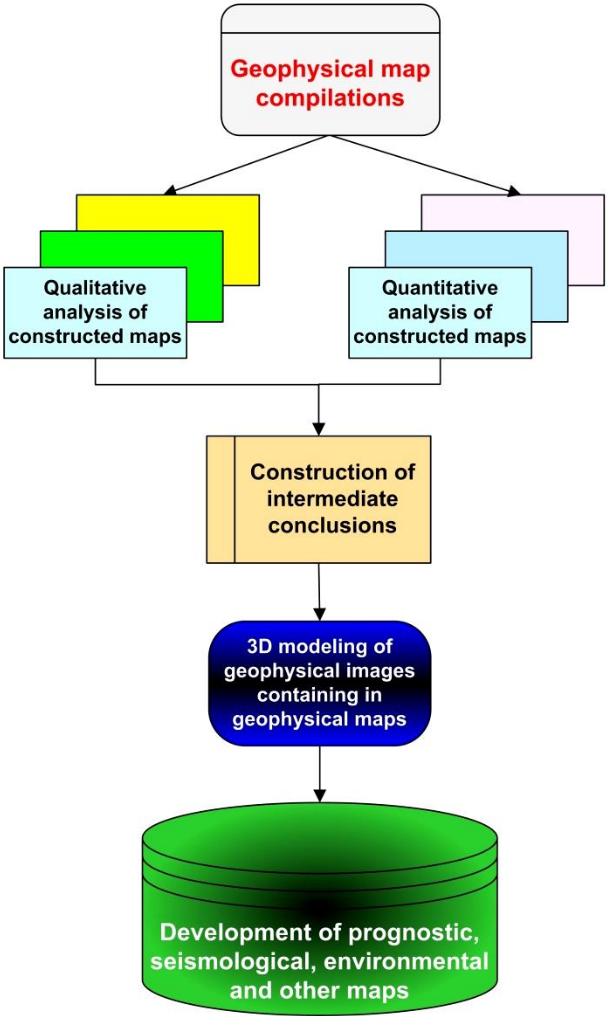

Figure 1: Simplified scheme of geophysical data processing.

Estimating the information value of geophysical and other means can be formalized on the basis of the following criteria [9]:

(1) Informativeness of the application (informational criterion Γ);

(2) Cost of implementing the method (cost criterion *C*);

(3) Time required to carry out the method (time criterion.

Criteria *C* and *T* are easy to calculate directly, but criterion Γ is a non-trivial research problem. A simplified algorithm can be written as:

$$\Omega = \Gamma \cup C \cup T, \qquad (1)$$

where ∪ is the symbol of unification.

All the available geophysical-geological information can be represented as the classic three-level model (Figure 2): (a) syntactic – quantity of information, (b) semantic – content of information, and (c) pragmatic – value of information. The logical-heuristic model for describing environmental information thus takes the following form:

$$\Gamma = I \cup R \cup V, \qquad (2)$$

where *I* is the quantitative estimation of information, *R* is the estimation of informational reliability corresponding to the semantic criterion, and *V* is the estimation of informational value in terms of feasibility according to the pragmatic criterion. Algorithm (2) is based on the fundamental terms of information theory and is combined with the structural (hierarchical) approach. This approach defines each indicator as a structure reflecting a set of typical situations and is then used to calculate the value of each estimator using the informational measure. Parameters *V* and *R* should be estimated geologically and by informational way, but there are beyond the scope of this paper. Here parameters *V* and *R* will be neglected, and it is assumed that $\Gamma = I$.

Probabilistic Approach to Geophysical-Geological Investigations

Necessary Math Background

Let us assume the following designations. Feature *R* is any independent characteristics of geological target: thickness, density, color, etc. Mean *S* is geophysical (geological) prospecting procedure providing information about the direct or circumstantial geological feature (features): geophysical data, drilling, geological mapping, geochemical analyses, etc.

Definition I: Beneath reliability of mean *S* with respect to feature *R* we will comprehend a probability of truth for hypothesis Γ: value of feature *R* equals to value obtaining by mean *S*.

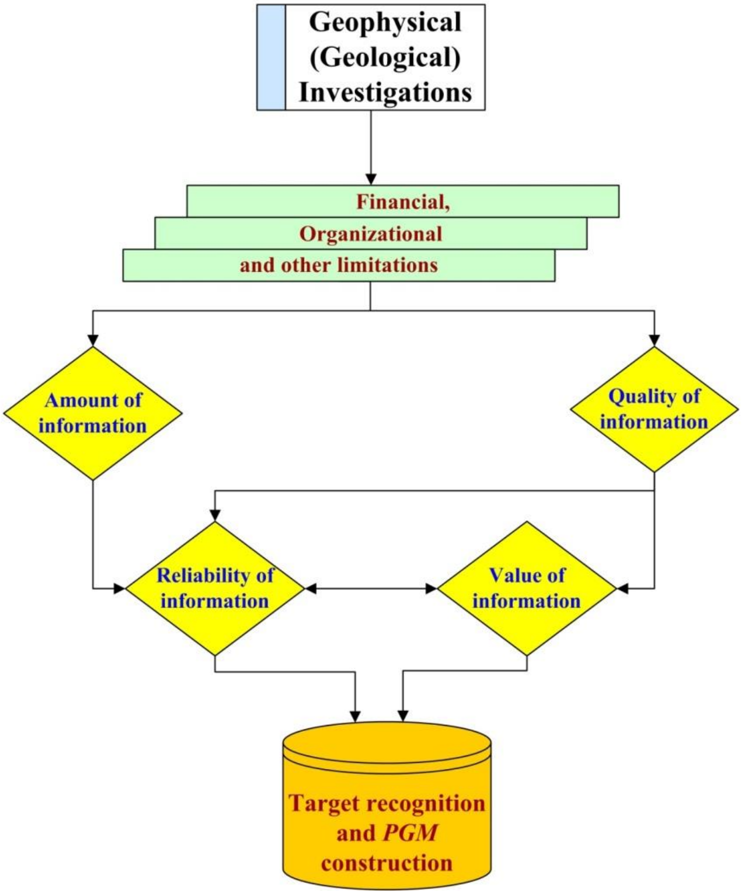

Figure 2: A scheme of desired geophysical target recognition and *PGM* construction with elements of information theory.

Thus, the main aim of geophysical prospection is to obtain the maximum effectiveness by minimum expenditures.

Symbol $S = \{S_i\}_{i=\overline{1,n}}$ will designate an arbitrary integration of the geological means.

If values of feature are measured on some numerical scale, we will consider this feature as quantitative, otherwise will believe that the feature as qualitative one. The main difference between the quantitative and qualitative means are that numerical scales of the quantitative features are ordered, at that time in a common case scales of qualitative parameters have no order.

Let us S and R are the fixed mean and feature, respectively and $\{r_1, r_2, \ldots, r_k\}$ is a set of values, which may include feature R. We will consider that result of local determination of feature R by mean S always includes some alternative r_τ, $1 \le r \le k$, which generally speaking, may differ from the real value of feature R.

If continuous scale of feature R has been divided to intervals, then determination of R reduces to finding the concrete interval to which pertains this feature. In this case we can consider that $r_1, r_2, \ldots, r_k$ are the points belonging to intervals of dividing (for instance, middles of these intervals). Obviously that among the values $r_1, r_2, \ldots, r_k$ is always such r_t, which belongs to the same interval that and real value of feature R. Then difference between the real value of feature and value r_t does not exceed a length of respective interval, and value r_t may be considered as real since we propose that dividing of scale for feature R is being with the necessary accuracy. Further under real value of feature R we will imply the mentioned value r_t.

Problem Statement

Introduced notion of reliability of mean S relative to feature R is a quantitative measure of frequency of coinciding feature R (obtained by use of mean S) with its real value.

A case when feature R is determining not by one mean S but by a set of means $S = \{S_i\}_{i=\overline{1,n}}$ has the following peculiarity. Any series of observations (investigations) realized by a set S is defined, obviously, not by one alternative r_τ, but a set of alternatives $r_{\tau_1}, r_{\tau_2}, \ldots, r_{\tau_n}$, which, broadly speaking, are different ones. In this case we cannot speak about one value of feature R generating by a set S. Then the following question is arising:

having a set of obtained values *R* - $\left(r_{\tau_1}, r_{\tau_2}, ..., r_{\tau_n}\right)$, which alternative r_τ needs to be recognized as a value of feature *R*. By other words, which *a priori* hypotheses from set *k* (the real value of *R* equals to r_t, $1 \le t \le k$) maybe assumed as the most suitable. Selection of the best (in a sense) hypothesis should be, evidently, realized by use of some algorithm (rule). Formally such a rule may be considered as mapping $\aleph$ of a set of possible indications of means $\left(r_{\tau_1}, r_{\tau_2}, ..., r_{\tau n}\right)$ to the set of values of features $\{r_t\}_{t=\overline{1,k}}$:

$$\aleph : \left\{\left(r_{\tau_1}, r_{\tau_2}, ..., r_{\tau_n}\right)\right\}_{(\tau_1, \tau_2, ..., \tau_n)} \to \{r_t\}_{t=\overline{1,k}} . \qquad (3)$$

If we will associate some fixed rule $\aleph = \aleph(S,R)$ with each feature *R* and set of means $S = \{S_i\}_{i=\overline{1,n}}$, then we will consider that *S* uniquely determines *R*. Really, series of local observations of feature *R* defines a set of alternatives $r_{\tau_1}, r_{\tau_2}, ..., r_{\tau_n}$, and rule $\aleph$ gives to the set one single value $\aleph\left(r_{\tau_1}, r_{\tau_2}, ..., r_{\tau_n}\right)$ of feature *R*.

Taking into account that the formalizability of problems associated with finding of reliability is mathematically complicated, we have no ways for an identical definition of the rule $\aleph$. In a common case, if $\aleph_1$ and $\aleph_2$ are two different rules, we could not find a simple method for their comparison. Proposed here probabilistic approach to definition of reliability allows to formulate criterion for comparison of the rules and to solve simultaneously a problem of the optimal selection (in a sense of the criterion) rule.

Let us suppose for definiteness that real value of feature *R* is r_1. Results of some *R* determination by some mean *S* may differ from r_1 because of the determination inaccuracy. Obviously, set of possible indications of the mean *S* may be described using some probability distribution [9]:

$$P_{11} = P\left(r_1^o \middle| r_1^r\right),\ P_{12} = P\left(r_2^o \middle| r_1^r\right),\ldots,\ P_{1k} = P\left(r_k^o \middle| r_1^r\right),$$

(4)

where $P_{1\tau} = P\left(r_\tau^o \middle| r_1^r\right)$ is the conditional probability of that results of R determination is r_τ, if the real value of feature R is r_1 (indexes "o" and "r" designate the "observed" and "real" values, respectively).

Generally speaking, value r_2 of feature R corresponds to another set of probabilities: P_{21}, P_{22}, …, P_{2k}. In the common case probabilities P_{t1}, P_{t2}, …, P_{tk} depend on t (i.e. on the real value of feature R). We will consider a matrix of conditional probabilities $\left\{P_{tr} = P\left(r_\tau^o \middle| r_t^r\right)\right\}_{t,\tau=\overline{1,k}}$ for each pair (S, R) by introducing expert methods since to obtain it by a logical way is practically impossible:

$$\Re(S,R) = \begin{pmatrix} P_{11} & \ldots & P_{1k} \\ \ldots & \ldots & \ldots \\ P_{k1} & \ldots & P_{kk} \end{pmatrix}. \qquad (5)$$

We must note that the matrix depends not only on S and R but also on the concrete physical-geological conditions of geological prospecting. However, relationship of $\Re$ (S, R) from different geological (physical, chemical, etc.) factors is not discussed here since we believe that these factors in real conditions are fixed. It may appear that obtaining matrix $\Re$ (S, R) containing k^2 numbers is practically insolvable problem. However, in real conditions number of independent elements in this matrix is greatly reduced.

It was mentioned above that probabilities $P_{t\tau}$ are conditional ones. $P_{tr} = P\left(r_\tau^o \middle| r_t^r\right)$ is the conditional probability of observed value r_τ of feature R by condition that r_τ is the real value of the feature. However, on the practice an inverse problem may have a vital importance: when using observed indication of the geological-geophysical mean, is necessary to give a probabilistic estimation of the real value. Mathematically this offers to the problem of determination of probabilities $\left\{\tilde{P}_{\tau t} = P\left(r_\tau^r \middle| r_t^o\right)\right\}_{t,\tau=\overline{1,k}}$. In contradiction to $P_{t\tau}$, indexes "o" and "r" here are interchanged.

Probabilities $\tilde{P}_{\tau t}$, generally speaking, do not associated with $P_{t\tau}$ by a hard analytical relationship. However, if *a priori* probabilities $P(r_t)$ of alternative values of R are known, then values $\tilde{P}_{\tau t}$ may be expressed through $P_{t\tau}$ using well-known Bayes's expression [10,11]:

$$\tilde{P}_{\tau t} = P\left(r_t^r \middle| r_\tau^o\right) = \frac{P\left(r_\tau^o \middle| r_t^r\right) P(r_t)}{\sum_{t=1}^{k} P\left(r_\tau^o \middle| r_t^r\right) P(r_t)} \quad . \tag{6}$$

If we have no any initial information about the values of geological-geophysical feature, we will consider that $P(r_1) = P(r_2) = ... = P(r_k) = 1/k$. In this case, Eq. (6) will be simplified:

$$\tilde{P}_{\tau t} = P\left(r_t^r \middle| r_\tau^o\right) = \frac{P\left(r_\tau^o \middle| r_t^r\right)}{\sum_{t=1}^{k} P\left(r_\tau^o \middle| r_t^r\right)} \quad . \tag{7}$$

Thus, if for all τ the following equality is fulfilled:

$$\sum_{t=1}^{k} P\left(r_\tau^o \middle| r_t^r\right) = 1, \tag{8}$$

then $P\left(r_\tau^o \middle| r_t^r\right) = P\left(r_t^r \middle| r_\tau^o\right)$. Equality (8) indicates that a sum of elements of any column of the matrix $\Re$ (S, R) = 1.

Further we will suppose that $P(r_1) = P(r_2) = ... = P(r_k) = 1/k$, i.e. alternatives r_1, r_2, …, r_k have *a priori* the equal probabilities. Here fulfilling of equality (8) is not obligatory one.

Reliability of Mean S Relative to Feature R

From the *definition* I and description of the total probability [10] follows that reliability of the mean *S* relative to the feature *R* may be calculated using the following formula

$$d(S;R)=\sum_{\tau=1}^{k}P\left(r_{\tau}^{o}\right)\cdot P\left(r_{\tau}^{r}\middle|r_{t}^{o}\right). \tag{9}$$

Taking into account that

$$P\left(r_{\tau}^{o}\right)=\sum_{\tau=1}^{k}P\left(r_{t}^{r}\right)P\left(r_{\tau}^{o}\middle|r_{t}^{r}\right)=\frac{1}{k}\sum_{t=1}^{k}P\left(r_{\tau}^{o}\middle|r_{t}^{r}\right)$$

and

$$P\left(r_{\tau}^{r}\middle|r_{\tau}^{o}\right)=\frac{P\left(r_{\tau}^{o}\middle|r_{\tau}^{r}\right)}{\sum_{t=1}^{k}P\left(r_{\tau}^{o}\middle|r_{t}^{r}\right)}\ ,$$

Eq. (9) has been transformed to the following form:

$$d(S;R)=\sum_{\tau=1}^{k}\left(\frac{1}{k}\sum_{t=1}^{k}P\left(r_{\tau}^{o}\middle|r_{t}^{r}\right)\frac{P\left(r_{\tau}^{o}\middle|r_{\tau}^{r}\right)}{\sum_{t=1}^{k}P\left(r_{\tau}^{o}\middle|r_{t}^{r}\right)}\right)=\frac{1}{k}\sum_{\tau=1}^{k}P\left(r_{\tau}^{o}\middle|r_{\tau}^{r}\right)=\frac{1}{k}\sum_{\tau=1}^{k}P_{\tau\tau}\ . \tag{10}$$

A few trivial examples are presented below.
Example 1:

$$\Re\left(S_1,R_1\right)=\begin{pmatrix}0.8 & 0.2\\ 0.3 & 0.7\end{pmatrix};\qquad d(S_1;R_1)=0.5(0.8+0.7)=0.75.$$

Example 2:

$$\Re\left(S_2,R_1\right)=\begin{pmatrix}0.9 & 0.1\\ 0.4 & 0.6\end{pmatrix};\qquad d(S_2;R_1)=0.5(0.9+0.6)=0.75.$$

Example 3:

$$\Re(S_3, R_2) = \begin{pmatrix} 0.8 & 0.1 & 0.1 \\ 0.1 & 0.8 & 0.1 \\ 0.1 & 0.1 & 0.8 \end{pmatrix}; \quad d(S_3; R_2) = 1/3(0.8 + 0.8 + 0.8) = 0.8.$$

In first two examples matrixes corresponding to the pairs (S_1, R_1) and (S_2, R_2) are different, but their probabilities are equal between themselves.

Reliability of the Set of Means S_i Relative to Feature R

Let us consider definition of feature *R* by set of means $S = \{S_i\}_{i=\overline{1,n}}$. The proposed methodology is based on realizing the following axiom [9]:

Axiom: A sequence of indications of means $S = \{S_i\}_{i=\overline{1,n}}$ (replacing feature *R*) is independent one. It means that

$$P\left(r_{\tau_1}^o, r_{\tau_2}^o, \ldots, r_{\tau_n}^o \middle| r_t^r\right) = P\left(r_{\tau_1}^o \middle| r_t^r\right) \cdot P\left(r_{\tau_2}^o \middle| r_t^r\right) \cdot \ldots \cdot P\left(r_{\tau_n}^o \middle| r_t^r\right). \quad (11)$$

As it was mentioned above, we are needed to agree, which value of *R* we will select as the most plausible hypothesis about the real value of *R* (by each fixed set of indications of means). A variant of the possible analysis of geological information is presented in Figure 2.

After series of observations of feature *R* (using a set of means $S = \{S_i\}_{i=\overline{1,n}}$), we will receive a set of alternatives $r_{\tau_1}^o, r_{\tau_2}^o, \ldots, r_{\tau_n}^o$. Which of the following *k* hypotheses Γ_t ($r_t^r = r_t^0$) we should adopt as the most plausible? Obviously, this should be hypothesis, for which the respective probability

$$P\left(r_t^r \middle| r_{\tau_1}^o, r_{\tau_2}^o \ldots, r_{\tau_n}^o\right) \quad (12)$$

will admit the maximum value.

Let us designate that ℵ* is mapping (rule) placing in requirement to each sequence of means such alternative value r_t* of feature *R*, on which the maximal value of Eq. (12) reaches.

Definition II: Reliability of the set of means *S* relative to the feature *R* is the probability of coinciding feature *R* (determined by the rule ℵ*) with the real value of *R*.

Taking into account *definition* II and expression of the total probability we have

$$d(S_1, S_2, ..., S_n; R) = \sum_{(\tau_1, \tau_2, ..., \tau_n)} P\left(r^o_{\tau_1}, r^o_{\tau_2}, ..., r^o_{\tau_n}\right) \cdot P\left(r^r_{t^*} \middle| r^o_{\tau_1}, ..., r^o_{\tau_n}\right). \quad (13)$$

Transforming Eq. (13) analogously to conversion of Eq. (9) to Eq. (10), we will receive the following expression for calculation of reliability of the set *S* relative to feature *R* :

$$d(s_1, s_2, ..., s_n; R) = \frac{1}{k} \sum_{(\tau_1, \tau_2, ..., \tau_n)} P_{t^*\tau_1} \cdot P_{t^*\tau_2} \cdot ... \cdot P_{t^*\tau_n} . \quad (14)$$

The rule ℵ* setting up a correspondence between the set of possible indications of the means $\left\{r^o_{\tau_1}, r^o_{\tau_2}, ..., r^o_{\tau_n}\right\}$ and the set of values of feature $\{r_t\}$, sets up simultaneously correspondence between the indexes:

$$\aleph^*: (\tau_1, \tau_2, ..., \tau_n) \rightarrow t^* . \quad (15)$$

It is supposed that *t** in Eq. (14) is defined from relationship (15) for the each fixed set $(\tau_1, \tau_2, ..., \tau_n)$.

Let us explain Eq. (14) by use of the following simplified example.

Example 4: Let us feature R is determined by means S_1 and S_2. It is necessary to calculate a reliability of means S_1 and S_2 relative to R (R may take values r_1 and r_2) by the given matrixes $\Re$ (S_1, R) and $\Re$ (S_2, R):

$$\Re(S_1, R) = \begin{pmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{pmatrix}, \quad \Re(S_2, R) = \begin{pmatrix} 0.9 & 0.1 \\ 0.4 & 0.6 \end{pmatrix}.$$

In this case a set of possible indications of means S_1 and S_2 consists of four elements:

$$r_1^o, r_1^o; \;\; r_1^o, r_2^o; \;\; r_2^o, r_1^o; \;\; r_2^o, r_2^o \;\; .$$

The relationships between probabilities of the means and the real values of feature R are compiled in Table 1 [9].

It easy to see that for the data (R, S_1, S_2) mapping (rule) $\aleph^*$ can be described by the following manner:

$$\left\langle \begin{array}{ll} r_1^o, r_1^o \to r_1^r, & \text{since}\; 0.8 \cdot 0.9 > 0.3 \cdot 0.4; \\ r_1^o, r_2^o \to r_2^r, & \text{since}\; 0.8 \cdot 0.1 < 0.3 \cdot 0.6; \\ r_2^o, r_1^o \to r_2^r, & \text{since}\; 0.2 \cdot 0.9 < 0.7 \cdot 0.4; \\ r_2^o, r_2^o \to r_2^r, & \text{since}\; 0.2 \cdot 0.1 < 0.7 \cdot 0.6. \end{array} \right\rangle$$

Table 1: Values of the conditional probabilities $P\left(r^o_{\tau_1}, r^o_{\tau_2} \middle| r^r_t\right)$.

Values of feature	Variants of the indications of means			
	$r^o_1 r^o_1$	$r^o_1 r^o_2$	$r^o_2 r^o_1$	$r^o_2 r^o_2$
r^r_1	0.8 · 0.9	0.8 · 0.1	0.2 · 0.9	0.2 · 0.1
r^r_2	0.3 · 0.4	0.3 · 0.6	0.7 · 0.4	0.7 · 0.6

Then reliability $d(S_1,S_2; R)$ may be determined as

$$d(S_1, S_2; R) = 0.5(0.8 \cdot 0.9 + 0.3 \cdot 0.6 + 0.7 \cdot 0.4 + 0.7 \cdot 0.6) = 0.8.$$

We must note that

$$\left.\begin{array}{l} d(S_1; R) = 0.5(0.8 + 0.7) = 0.75 \\ d(S_2; R) = 0.5(0.9 + 0.6) = 0.75 \end{array}\right\} .$$

In the inspected case

$$d(S_1, S_2; R) > d(S_1; R) = d(S_2; R). \qquad (16)$$

It can be shown that the non-strict inequalities analogical to Eq. (16) will be fulfilled always for any set of means $S = \{S_i\}_{i=\overline{1,n}}$ and for any matrixes $\Re$ (S_1, R). This condition will be realized in the case if probability $P_{\tau\tau}$, $1 \le \tau \le k$ in the line with number τ in any matrix will have the most value. The last requirement has the following significance: the most probable variant for each indication of mean is an output corresponding to the real value of feature R. Thus, inequality

$$d(S_1, S_2, ..., S_n; R) \ge \max_{i=\overline{1,n}} \{d(S_i; R)\} \qquad (17)$$

is realizing for any R and $\{S_i\}_{i=\overline{1,n}}$.

At the same time, left part in Eq. (17) left part may equal to the right part (by definite R and $\{S_i\}_{i=\overline{1,n}}$). Such a variant will have place when each of the matrixes $\Re$ (S_1, R) and $\Re$ (S_2, R) has the following form

$$\begin{pmatrix} P & 1-P \\ 1-P & P \end{pmatrix},$$

i.e. they are symmetrical.

Example 5: Let us

$$\Re(S_1, R) = \begin{pmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{pmatrix}, \quad \Re(S_2, R) = \begin{pmatrix} 0.7 & 0.2 \\ 0.2 & 0.7 \end{pmatrix}.$$

In this case the table of the conditional probabilities $P\left(r_{\tau_1}^o, r_{\tau_2}^o \middle| r_t^r\right)$ will have the following form:

Table 2: Values of the conditional probabilities for *Example 5*.

Values of feature	Variants of the indications of means			
	$r_1^o r_1^o$	$r_1^o r_2^o$	$r_2^o r_1^o$	$r_2^o r_2^o$
r_1^r	0.8 · 0.7	0.8 · 0.3	0.2 · 0.7	0.2 · 0.3
r_2^r	0.2 · 0.3	0.2 · 0.7	0.8 · 0.3	0.8 · 0.7

The rule $\aleph$* in this case maybe written as

$$\left\langle \begin{array}{ll} r_1^o, r_1^o \to r_1^r, & \text{since}\, 0.8\cdot 0.7 > 0.2\cdot 0.3; \\ r_1^o, r_2^o \to r_2^r, & \text{since}\, 0.8\cdot 0.3 < 0.2\cdot 0.7; \\ r_2^o, r_1^o \to r_2^r, & \text{since}\, 0.2\cdot 0.7 < 0.8\cdot 0.3; \\ r_2^o, r_2^o \to r_2^r, & \text{since}\, 0.2\cdot 0.3 < 0.8\cdot 0.7 \end{array} \right\rangle.$$

Correspondingly,

$d(S_1, S_2; R) = 0.5(0.56 + 0.24 + 0.56 + 0.24) = 0.8$.

In as much as $d(S_1; R) = 0.5(0.8 + 0.8) = 0.8$ and $d(S_1; R) = 0.5(0.7 + 0.7) = 0.7$, then for the considered example in Eq. (17) we have a sign of equality.

It is also important that values $d(S_i; R)$, $1 \le i \le n$ in the common case do not allow to determining identically the value $d(S_1, S_2, ..., S_n; R)$. For instance, if in *Example 4* we will replace matrix $\Re$ (S_2, R) by the matrix

$$\Re(S_3, R) = \begin{pmatrix} 0.6 & 0.4 \\ 0.1 & 0.9 \end{pmatrix},$$

then for a new pair of matrixes $\Re$ (S_1, R) and $\Re$ (S_3, R) we will have: d (S_1, R) = d (S_3, R) = 0.75 (as in the previous example), but d (S_1, S_3; R) = 0.5 (0.48+0.32+0.12+0.63) = 0.775.

Thus, reliabilities of means S_1, S_2 and S_3 relative to feature R are equal, because d (S_1; R) = d (S_2; R) = d (S_3; R) = 0.75. However, reliability of pair of the means (S_1, S_2) relative to R is higher that reliability of the pair (S_1, S_3):

$d(S_1, S_2; R) = 0.8 > d(S_1, S_3; R) = 0.775.$

Let us introduce a new notion "increment of reliability" $\Delta d^I(S_1, S_2, ..., S_n; R)$ for each R and set of means $S = \{S_i\}_{i=\overline{1,n}}$.

This value may be defined using the following expression:

$$\Delta d^{I}\left(S_1, S_2, ..., S_n; R\right) = d\left(S_1, S_2, ..., S_n; R\right) - \max\left\{d\left(S_i, R\right)\right\}, \quad 1 \le i \le n.$$

(17)

Introduced value $\Delta d^{I}\left(S_1, S_2, ..., S_n; R\right)$ has the following significance. It shows how much the reliability of $S = \left\{S_i\right\}_{i=\overline{1,n}}$ exceeds the reliability of the most precise mean from the set S_i.
We could also consider a new parameter Δd^{II} :

$$\Delta d^{II}\left(S_1, S_2, ..., S_n; R\right) = \left(S_1, S_2, ..., S_n; R\right) - \max_{1 \le i_1,\ i_2 \le n}\left\{d\left(S_{i_1}, S_{i_2}; R\right)\right\},$$

(18)

which defines how much the reliability of set $S = \left\{S_i\right\}_{i=\overline{1,n}}$

exceeds the reliability of the most precise pair of means (in terms of $\max d\left(S_{i_1}, S_{i_2}, R\right)$), involving to the set *S*.

Analogically we can define $\Delta d^{III}\left(S_1, S_2, ..., S_n; R\right)$,

$\Delta d^{IV}\left(S_1, S_2, ..., S_n; R\right)$ and so on.

For the above-mentioned *Example 4* we have:

$\Delta d^{I}\left(S_1, S_2; R\right) = 0.8 - 0.75 = 0.05; \quad \Delta d^{I}\left(S_1, S_3; R\right) = 0.025.$

For *Example 5* $d\left(S_1, S_2; R\right) = d\left(S_1; R\right)$, consequently

$d\left(S_1, S_2; R\right) = d\left(S_1; R\right)$ and $d\left(S_1, S_2; R\right) = \max_{i=1,2}\left\{d\left(S_i; R\right)\right\}$.

Therefore, $\Delta d^{I}\left(S_1, S_2; R\right) = 0.$

The last result does not contradict to logics, as it may appear for the first view. If we will again refer to the rule ℵ* constructed for *Example 5*, we could see that the values of feature R determined by the rule $\aleph^*$ precisely correspond to indications of first mean S_i. Then and reliability of the pair of means (S_1, S_2) relative to feature R will be equal to reliability of the mean S_1. Therefore, here is realized an equality $d(S_1,S_2;R)=d(S_1;R)$, whence $\Delta d^I(S_1,S_2;R)=0$ [9].

In *Example 4* the described above situation has no place, therefore corresponding value $\Delta d^I(S_1,S_2;R)$ is positive one.

We will examine additional example where number of means n = 3.

Example 6: Let us

$$\Re(S_1,R)=\Re(S_2,R)=\Re(S_3,R)=\begin{pmatrix}0.8 & 0.2\\ 0.2 & 0.8\end{pmatrix}.$$

It is easy to show that

$$\left.\begin{array}{l} d(S_1,S_2;R)=d(S_1,S_3;R)=d(S_2,S_3;R)=0.8\\ \text{and}\\ \Delta d^I(S_1,S_2;R)=\Delta d^I(S_1,S_3;R)=\Delta d^I(S_2,S_3;R)=0.\end{array}\right\}$$

The necessary data for calculation of value $d(S_1,S_2,S_3;R)$ are compiled in Table 3.

Table 3: Values of the conditional probabilities $P\left(r^o_{\tau_1},r^o_{\tau_2},r^o_{\tau_3}\middle|r^r_t\right)$.

Values of feature	Variants of the indications of means			
	$r^o_1 r^o_1 r^o_1$	$r^o_1 r^o_1 r^o_2$	$r^o_1 r^o_2 r^o_1$	$r^o_1 r^o_2 r^o_2$
r^r_1	0.8 · 0.8 · 0.8	0.8 · 0.8 · 0.2	0.8 · 0.2 · 0.8	0.8 · 0.2 · 0.2
r^r_2	0.2 · 0.2 · 0.2	0.2 · 0.2 · 0.8	0.2 · 0.8 · 0.2	0.2 · 0.8 · 0.8
	$r^o_2 r^o_1 r^o_1$	$r^o_2 r^o_1 r^o_2$	$r^o_2 r^o_2 r^o_1$	$r^o_2 r^o_2 r^o_2$
r^r_1	0.2 · 0.8 · 0.8	0.2 · 0.8 · 0.2	0.2 · 0.2 · 0.8	0.2 · 0.2 · 0.2

r_2^r	0.8 · 0.2 · 0.2	0.8 · 0.2 · 0.8	0.8 · 0.8 · 0.2	0.8 · 0.8 · 0.8

The rule ℵ* in this case can be presented as:

$$\left\langle \begin{array}{ll} r_1^o, r_1^o, r_1^o \to r_1^r, & \text{since } 0.8 \cdot 0.8 \cdot 0.8 > 0.2 \cdot 0.2 \cdot 0.2; \\ r_1^o, r_1^o, r_2^o \to r_1^r, & \text{since } 0.8 \cdot 0.8 \cdot 0.2 > 0.2 \cdot 0.2 \cdot 0.8; \\ r_1^o, r_2^o, r_1^o \to r_1^r, & \text{since } 0.8 \cdot 0.2 \cdot 0.8 > 0.2 \cdot 0.8 \cdot 0.2; \\ r_1^o, r_2^o, r_2^o \to r_2^r, & \text{since } 0.8 \cdot 0.2 \cdot 0.2 < 0.2 \cdot 0.8 \cdot 0.8; \\ r_1^o, r_1^o, r_1^o \to r_1^r, & \text{since } 0.2 \cdot 0.8 \cdot 0.8 > 0.8 \cdot 0.2 \cdot 0.2; \\ r_1^o, r_1^o, r_2^o \to r_2^r, & \text{since } 0.2 \cdot 0.8 \cdot 0.2 < 0.8 \cdot 0.2 \cdot 0.8; \\ r_2^o, r_2^o, r_1^o \to r_2^r, & \text{since } 0.2 \cdot 0.2 \cdot 0.8 < 0.8 \cdot 0.8 \cdot 0.2; \\ r_2^o, r_2^o, r_2^o \to r_2^r, & \text{since } 0.2 \cdot 0.2 \cdot 0.2 < 0.8 \cdot 0.8 \cdot 0.8 \end{array} \right\rangle.$$

Taking into account the data presented in Table 3 and the rule ℵ*, value

$$d(S_1, S_2, S_3; R) = 0.5(2 \cdot 0.8 \cdot 0.8 \cdot 0.8 + 6 \cdot 0.2 \cdot 0.8 \cdot 0.8) = 0.896,$$

whence

$$\Delta d^I(S_1, S_2, S_3; R) = 0.896 - 0.8 = 0.096.$$

If we will introduce to the above considered example

$$\Re(S_1, R) = \Re(S_2, R) = \Re(S_3, R) = \begin{pmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{pmatrix},$$

we obtain

$$d(S_1, S_2, S_3; R) = 0.5(2 \cdot 0.7 \cdot 0.7 \cdot 0.7 + 6 \cdot 0.3 \cdot 0.7 \cdot 0.7) = 0.784,$$

and

$$\Delta d^I(S_1, S_2, S_3; R) = 0.784 - 0.7 = 0.084.$$

Evaluating the Efficiency of Geophysical Methods with Informational-Statistical Procedures

Choosing the right method (or number of methods) can be based on a quantitative estimate (Figure 2). For this purpose, reliable informational and statistical criteria are needed. The first issue is the quantity of information that can be obtained by a single method or a set of methods. The second is to define a criterion to express the decision-making risk as a function of the geophysical data. Nevertheless, informational criteria are preferable, because geophysical prospecting is a permanent process of acquisition and analysis of information [9].

The classic information theoretic works by Shannon [12] and Brillouin [13] prompted Khalfin [14] to apply these criteria to geophysics. It has been shown [15,16] that informational and statistical approaches represent two aspects of a shared approach. For instance, the solution to an identification problem using the criterion of minimal average risk or that of maximum information (minimal residual uncertainty) under certain conditions results in the same expressions [17,18].

The applied geophysical fields usually have maximal and minimal intensity within studied areas. The difference between the maximal and minimal intensities may be subdivided into some intervals (gradations). Gradations of indicators can be also used to obtain information about the types of the desired objects. Physical fields and their transformations, geochemical analyses, some geological features, etc. can serve as possible indicators.

Some General Considerations

Here are considered some key elements of the theory of information conformably to geophysical studies. Illustrating how these methods can yield valuable results should highlight commonalities in geophysical theory and hence optimize *PGM* development and successful solving geological-geophysical problems.

As shown in countless publications on geophysical prospecting [e.g., 6-8,19,20], there are typically no more than two or three geophysical methods employed to solve problems of detection, contouring and the development of 3-D models of the objects under study. Let us consider the results of the application of two methods (e.g., gravity and magnetic surveys). They are designated in Table 4 as follows: 1 = negative field, 2 = positive field, 3 = roughly zero field, 4 = alternating field, and 5 = high-gradient field.

Four combinations of two parameters can represent four classes of desired targets, each ranging from 1 to 5. The number of possible combinations of two parameters divided into five categories is 25. The geological features (each class) can be characterized by one of these 25 combinations. It is also necessary to take into account concrete geological situation. For example, in Table 4 value 11 can be obtained for aqueous marl (instead of gabbroid), and this may lead to an erroneous conclusion in the process of geological-geophysical interpretation. Therefore, a certain abundance of the set of geophysical methods sometimes is advisable.

Table 4: An example of integrated interpretation.

Typical field combination										Class of targets
Magnetic					Gravity					
1	2		4		1	2				
	♦					♦				Marl
♦					♦					Gabbroid
			♦						♦	Salt
	♦					♦				fracture zone

Each class of geological features can be characterized by one of these 25 combinations. The number of combinations can be increased at the expense of secondary parameters arising due to certain transformations of the fields (e.g., downward and upward continuation, various derivatives). Thus, measurements of two geophysical fields can provide sufficient data to resolve geological-geophysical mapping problems.

Let us assume that there is only one anomalous target (*AT*) in the area. This area is divided into *N* equal cells. For simplicity each cell is assumed to have the same probability of containing the *AT*. Thus, the probability of finding the *AT* is equal to $P = 1/N$ in each cell. Hence, the entropy of experiment β (discovery of *AT*) is $\log N$. The entropy is determined using the following expression:

$$H(\beta) = -\sum_{i=1}^{N} P(B_i) \cdot \log P(B_i), \tag{19}$$

where $P(B_i)$ is the probability of a B_i outcome (*B* is the range of outcomes of experiment β).

In this situation $H(\beta)$ value ($\log N$) is maximum possible uncertainty. Experiment α (geophysical observation) gives additional information. The *A* value is a range of outcomes for the experiment α.
The difference between uncertainties in the β results before and after experiment α serves to estimate the information in α as related to β:

$$I(\alpha, \beta) = H(\beta) - H(\beta|\alpha), \tag{20}$$

where $H(\beta|\alpha)$ is the conditional entropy for experiment β (provided that experiment α has been conducted).
The conditional entropy is the average value of a random variable taking a $H(\beta_i|\alpha_i)$ value with a probability of $P(A_i)$:

$$H(\beta_i|\alpha_i) = \sum_{i=1}^{N} P(A_i) H(B_i|A_i). \tag{21}$$

Estimating the Efficiency of Individual Geophysical Methods

When selecting methods for integration, it makes sense to evaluate the amount of information provided by each geophysical method.

Starting from a well-investigated site (with an equal distance between observation points) typical of the area under study, it is assumed that 1/50 of it contains an *AT*. It is known that in the *AT* part the magnetic field is always positive, whereas in the empty part of the area it may be either positive or negative with equal probability. In other words, it is known *a- priori* that 2% and 98% of the area are target-containing and empty, respectively, and in 49% and 51% of this area the magnetic fields, respectively, are negative and positive. The results of experiment β can be designated as follows: $B - AT$ occupying part of the area, $\overline{B}$ – empty part of the area. Thus, $P(B) = 0.02$, $P(\overline{B}) = 0.98$. According to expression (19), $H(\beta) \cong 0.14$.

The result of experiment α is expressed as follows: A is a positive field, $\overline{A}$ is a negative field. The relative partial entropy (after recording the positive magnetic field at the measurement point) can be calculated in the following way: $P(A) = 0.51$; $P(\overline{A}) = 0.49$; $P(B|A) = 2/51$, $P(\overline{B}|A) = 1 - P(B|A) = 49/51$. Thus, $H(\beta|\alpha) \cong 0.24$. According to the recorded negative magnetic field, the area is certainly empty: $H(B|\overline{A}) = 0$. Consequently, the partial entropy for experiment β under the conditions of α, as stated in Eq. (21), is $H(\beta|\alpha) \cong 0.12$. Thus, the uncertainty of the determination of *AT* decreases after magnetic field measurement from 0.14 to 0.12.

Advantages of Geophysical Method Integration

What information can be obtained from geophysical field measurements? Assume that a magnetic field is observed in a range D, and the measurement precision is given by τ. The next step is to fit an integer number of small intervals ξ into the intervals D and τ. It is known after measurement that the value of the field in ξ units (with a precision up to ξ) fits into the interval τ. Using Eq. (20) and considering that the entropy in this expression before and after measurement is expressed by logarithms D/ξ and τ/ξ, respectively, one easily obtains (on the basis of [21]):

$$I(\alpha, \beta)_{\xi} = \log\left(\frac{D}{\tau}\right). \qquad (22)$$

With decreasing ξ, these entropies increase unlimitedly, but the information remains unchanged. With unrestricted increase in measurement accuracy the information also increases unlimitedly, but slowly: an n-fold increase in accuracy leads to only $\log N$ of information units.

An essential property of composite experiments is as follows. If certain tests ψ, φ and θ are independent, experiments ψ and θ can have zero information about φ. However, an integrated experiment ψ and θ can completely determine the outcome of experiment φ. Consequently, whereas separate geophysical methods give no information on the presence (or absence) of a target, this can be obtained by integrating these methods. It follows from Eq. (22) that the development of integrated investigations is more effective than increasing the accuracy of separate methods.

Estimates of the Efficiency of Geophysical Integration based on the Probability of type I and type II Errors

Classification efficiency can be estimated quantitatively not only for separate methods, but also for geophysical integration by calculating the reliability of revealing an *AT*. Let us assume that in some region 20 anomalies have been contoured and localized by a set of geophysical methods. The revealed anomalies were divided into 3 groups with various degrees of desired *AT* discovery. Under the assumption that the results of drilling are absolutely reliable, the classification reliability can be assessed by calculating the probability of type I and type II errors.

The probability of a type II error (M_2) is expressed as the relative frequency of an erroneous diagnosis for objects from sampling *B* (*AT*). The probability of a type I error (M_1) is expressed as the relative frequency of an erroneous diagnosis for objects from the sampling $\overline{B}$ (the remainder of the objects). These errors are used to determine the total unconditional error of separation *q* between classes *B* and $\overline{B}$ (i.e. the risk of an erroneous solution):

$$q = M_2 P(B) + M_1 P(\overline{B}), \tag{23}$$

where $P(B)$ and $P(\overline{B})$ are the prior probabilities of the appearance of objects of the first and second classes, respectively. If $P(B)$ = $P(\overline{B})$ = 0.5, then the *q* value corresponds to the intersection area of the distribution densities $P(X|B)$ and $P(X|\overline{B})$. Here *X* is the separation index. It can represent geophysical field amplitude or gradient, the value of integrated indicator, etc.

The separation reliability (γ) is:

$$\gamma = 1 - q. \tag{24}$$

The total empirical error should be compared to the theoretical error. The approximation of the errors can confirm a correct assumption and provide high reliability of identification. Using logical-informational methods [4], the classification reliability is estimated solely from empirical errors.

The errors due to assigning observation results to a class (with or without the *AT*) can be determined as follows. The absence of anomalies for the complex indicator in a known target-containing area is a type II error, or "omission of target". The presence of these anomalies in the empty part of this area is a type I error, or a "false alarm".

Comparing these rapid results with those of a more complex integration can also be used to estimate the respective errors and the reliability of classification. Let us assume that the hydrocarbon nature of 14 out of 20 recognized geophysical anomalies was confirmed. New *AT* were not revealed in the areas where geophysical methods were applied. Thus, $M_1 = 0$, $M_2 = 14/20$. Assuming that $P(B) = P(\overline{B}) = 0.5$, and taking into account Eqs. (23) and (24), then $q = 0.70$ and $\gamma = 0.30$.

Theory of Graphs

Complexity of Geophysical Method Integration

How many geophysical methods should be applied for searching economic deposits, localization of archaeological remains or revealing some dangerous geological phenomena (e.g., karst terrane)? Extending a set of methods could be considered to be at variance with its economic efficiency, and complicated from both an organizational and a technical point of view. In addition, there is a basic limitation imposed on the number of methods. As noted Duda and Hart [22], a growing number of target indicators require larger amounts of standard information. However, sufficient standards are only available in well-explored areas, where quantitative prediction is obviously less urgent. Therefore, a survey set should involve the minimum number of methods.

"Four Colors Theorem": Some Brief Information

Let us now examine the "four colors theorem" from this standpoint. Using elementary notions of graph theory the problem can be formulated as follows: prove that all vertices of an arbitrary planar graph can be colored with four colors in such a way that no two vertices joined by a common edge are the same color. It was proved as early as the middle of the 19th century that four colors suffice to color different counties on the map of England. However, a solution to this theorem was only found more than 100 years later [23,24]. The authors subdivided all possible maps into almost 2,000 types and developed a computer program for their investigation. For each type the problem – whether a map which cannot be colored with four colors can be found among the variety of maps – was solved. After lengthy investigations, an answer of “no” was obtained for all types, and this fact confirms the above solution.

A new (general) proof of the theorem was put forward recently [25]. Figure 3 shows seventeen of 633 reducible configurations which are displayed using the indicated convention. The whole set can be found in [25].

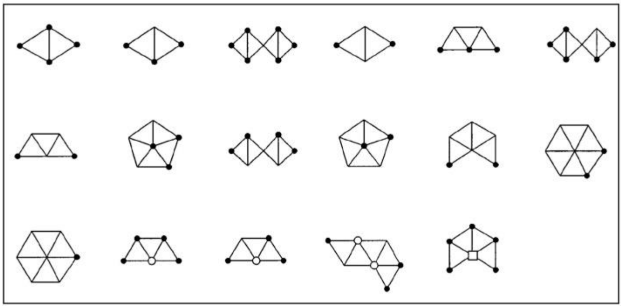

Figure 3: An example of possible configurations (after Appel and Haken, [23]).

Minimizing the Number of Combined Geophysical Methods by the “Four Colors Theorem”

The intuitive use of a small number of integration elements in practice can be theoretically substantiated applying the well-known mathematical and cartographic “four colors” solution [23-26] for integrating geophysical methods by solving different geological, environmental and other problems.

Geophysical investigation is usually a multistage procedure and for simplicity it is assumed that the goal of each prospecting stage is the selection of an area for more detailed operations at the next stage. The result of prospecting is primarily a substantiated evaluation of the areas under investigation and their classification into two groups: those worthy and unworthy of further study. The objective of prospecting is to obtain the maximum information at a given cost.

Any area under study can be divided into separate sub-areas according to certain indicators. The following system of prospect classification has been adopted in the USA [27]: high (*H*), medium (*M*), low (*L*) and unknown (*U*). The objective is to single out promising areas (if any) from the entire set by an integrated geophysical survey. The colors refer to different combinations of geophysical methods. A positive conclusion for a certain prospecting method is labeled (**+**), and a negative (–).

$$H_i(x,y,z,t)=\begin{Bmatrix} - & |H_i|<\varphi \\ + & |H_i|>\varphi \end{Bmatrix}, \qquad (25)$$

where φ is some assumed value indicating the split between negative and positive values of geophysical field H_i, x, y, z are the space coordinates and t is the time. On the right side of Eq. (25), H_i is assigned an absolute value since often anomalous targets may be reflected by negative geophysical anomalies.

Clearly, a combination of at least two independent geophysical methods is necessary for the first three gradations (*H*, *M* and *L*); gradation *U* implies no application of the method set (on a scale or not at all) in the area under investigation (Table 5).

The geophysical methods employed are *a priori* assumed to be of equal significance. The threshold field values (the split between plus and minus) and specific types of geophysical investigations are determined according to the prospecting results for similar objects investigated previously and other geological and geophysical considerations. The split refers to the threshold for field values representing specified physical characteristics. These physical characteristics may, for example, include amplitudes of observed fields, field gradients or indicators of field variability. The necessary condition is that we assume the geophysical anomalies are produced by the same targets [28].

Table 5: Subdivision of an area according to geophysical survey results.

Level of knowledge of the area	Geophysical method		Combination number (color)
	First	Second	
High (*H*)	+	+	1
Medium (*M*)	+	–	2
Low (*L*)	–	–	3
Unknown (*U*)	No necessary data		4

Let us demonstrate an applicability of this approach on a simple example. In an area of porphyry copper deposit in Kazakhstan (Figure 4), marginal values of electrical resistivity of 50 Ohm·m, and magnetic field of 0 nanoTesla have been selected (deposits of this type are characterized by low resistivity (< 50 Ohm·m) and negative magnetic field) [29]. Figure 4a shows isolines of resistivity, 3b – isolines of magnetic field, and 3c – coloring in four colors according to Table 5. The contour of the deposit is clearly revealed on the basis of the data of two methods.

It can be concluded from the foregoing that an optimum geophysical set consist of two independent geophysical methods. A map of geophysical results colored with four colors by the above technique serves as a basis for more detailed investigation. In this connection, a certain redundancy of the set is needed. It should, however, be kept in mind that the employed geophysical set is usually oriented to a particular problem and substantiated by a corresponding physical-geological model of the medium. Any change in the problem (e.g. an increase in the necessary depth of investigation) or in geological and geophysical pattern of the area may bring about a change in the set of methods. In this context, this division of the theory of graphs can be attributed to information theory.

At the same time, change of geophysical methods may, in turn, affect the “coloring” of the area under study [28,30]. Therefore, it is generally a good practice to use three geophysical methods which are effective under given conditions. For instance, a radiometric channel combined with aerial magnetic and electromagnetic surveys will give a negligible increase in cost, but increasing in informativeness may be significant one.

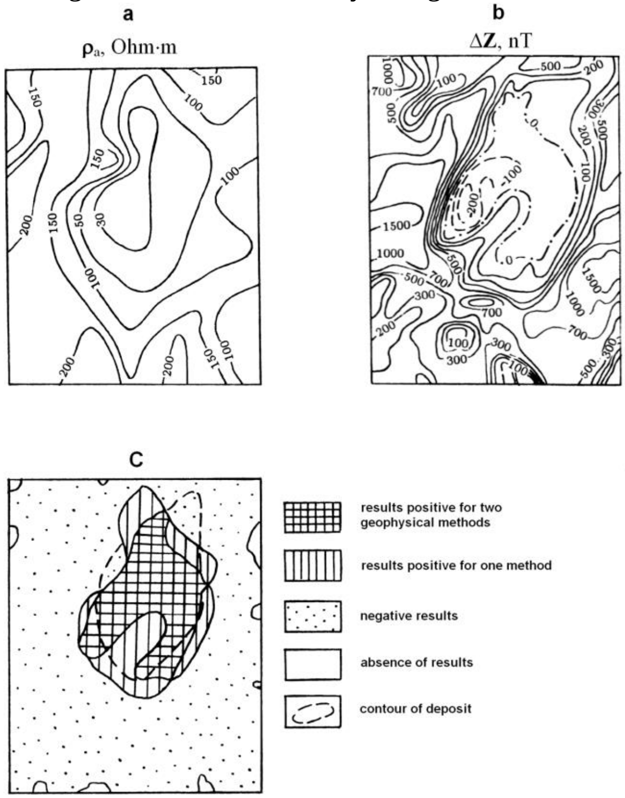

Figure 4: Electric (a) and magnetic (b) prospecting results at Benkaly porphyry copper deposit (Kazakhstan) and their processing according to four colors problem solution (c) (a, b and contour of deposit from [29]).

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